

TWISTOR INTERPRETATION OF SLICE REGULAR FUNCTIONS

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ABSTRACT. We study a geometric interpretation of the theory of quaternionic slice regular functions. The main idea come from a recent work by G.Gentili, S.Salamon and C. Stoppato [14], in which the authors found that any slice regular function can be lifted to a holomorphic curve in the space of twistors of \mathbb{S}^4 . In this paper we analyze some aspect of this lift and, in particular, we point out the projective classes of surfaces up to degree three, contained in \mathbb{CP}^3 , that fit in this frame. An explicit example is described at the end.

1. INTRODUCTION

In this paper we study the relation between orthogonal complex structures on subdomains of \mathbb{R}^4 and the recent theory of quaternionic slice regular functions.

Given a $2n$ -dimensional oriented Riemannian manifold (Ω, g) , an *almost complex structure* over Ω is an endomorphism $J : T\Omega \rightarrow T\Omega$, defined over the tangent bundle, such that $J^2 = -id$. An almost complex structure is said to be a *complex structure* if J is *integrable*, meaning, for instance, that the associated *Nijenhuis* tensor,

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY],$$

vanishes everywhere for each couple of tangent vectors X and Y ; it is said to be *orthogonal* if it preserves the Riemannian product, i.e. $g(JX, JY) = g(X, Y)$ for each couple of tangent vectors X and Y and preserves the orientation of Ω . Collecting everything, an *orthogonal complex structure* (OCS) is an almost complex structure which is integrable and orthogonal.

The condition for J to be an OCS depends only on the conformal class of g , so, if Ω is a four dimensional open subset of \mathbb{R}^4 endowed with the Euclidean metric g_{Eucl} , then the resulting theory is invariant under the group $SO(5, 1)$ of conformal automorphisms of $\mathbb{R}^4 \cup \{\infty\} \simeq \mathbb{H} \cup \{\infty\} \simeq \mathbb{S}^4$.

For any open subset Ω of \mathbb{R}^4 it is possible to construct standard OCSes, called *constant*, in the following way: think \mathbb{R}^4 as the space of real quaternions \mathbb{H} and fix any element $q \in \mathbb{S} := \{x \in \mathbb{H} \mid x^2 = -1\}$. Identifying each tangent space $T_p\Omega$ with \mathbb{H} himself, we define the complex structure \mathbb{J}_q everywhere by left multiplication by q , i.e.: $\mathbb{J}_q(p)v = qv$. Any OCS defined globally on \mathbb{H} is known to be constant (see Proposition 6.6 in [31]), moreover it was proven in [26] the following result.

Theorem 1 ([26], Theorem 1.3). *Let J be an OCS of class \mathcal{C}^1 on $\mathbb{R}^4 \setminus \Lambda$, where Λ is a closed set of zero 1-dimensional Hausdorff measure. Then J is the push-forward of the standard constant OCS on \mathbb{R}^4 under a conformal transformation and J can be maximally extended to the complement of a point $\mathbb{R}^4 \setminus \{p\}$.*

In the same paper it was proven the following result which completely solve the situation in a very particular case.

Theorem 2 ([26], Theorem 1.6). *Let \mathbb{J} be an OCS of class \mathcal{C}^1 on $\mathbb{R}^4 \setminus \Lambda$, where Λ is a round circle or a straight line, and assume that \mathbb{J} is not conformally equivalent to a constant OCS. Then \mathbb{J} is unique up to sign, and $\mathbb{R}^4 \setminus \Lambda$ is a maximal domain of definition for \mathbb{J} .*

In the assumption of Theorem 2 it is possible to construct explicitly the OCS \mathbb{J} as follows. Under the identification $\mathbb{R}^4 \setminus \Lambda \simeq \mathbb{H} \setminus \mathbb{R}$, a point x can be written as $x = x_0 + x_1i + x_2j + x_3k$ (with

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the usual multiplication rules of quaternions), or as $x = \alpha + I_x\beta$, where $\alpha = x_0$ is the real part of x , $I_x = (x_1i + x_2j + x_3k)/\sqrt{x_1^2 + x_2^2 + x_3^2} \in \mathbb{S}$ and $\beta = \sqrt{x_1^2 + x_2^2 + x_3^2}$, so that $I_x\beta$ represent the imaginary part of x . Then, for each $x = \alpha + I_x\beta \in \mathbb{H} \setminus \mathbb{R}$ we define \mathbb{J} such as $\mathbb{J}(x)v = I_xv$, for each $v \in T_x(\mathbb{H} \setminus \mathbb{R})$. This is an OCS over $\mathbb{H} \setminus \mathbb{R}$ that is constant on every complex line

$$\mathbb{C}_I := \{\alpha + I\beta \mid \alpha, \beta \in \mathbb{R}\}, \quad I \in \mathbb{S},$$

but not globally constant, hence $\pm\mathbb{J}$ are the only non-constant OCSes on this manifold (up to conformal transformations).

In [14] the authors proposed a new way to study the problem when $\Lambda \subset \mathbb{R}^4$ is a closed set of different type. The idea is to take the OCS \mathbb{J} , previously defined, and to push it forward on the set we are interested in. To do this we need to be sure that the function f , considered to push forward, preserves the properties of \mathbb{J} . Well, if the function f is a quaternionic slice regular function defined on a domain $\Omega \subset \mathbb{H}$ such that $\Omega \cap \mathbb{R} \neq \emptyset$, then this is true.

Definition 1. Let $\Omega \subset \mathbb{H}$ be a domain such that $\Omega \cap \mathbb{R} \neq \emptyset$ and consider a function $f : \Omega \rightarrow \mathbb{H}$. For any $I \in \mathbb{S}$ we use the following notation: $\Omega_I := \Omega \cap \mathbb{C}_I$ and $f_I := f|_{\Omega_I}$. The function f is called *slice regular* if, for each $I \in \mathbb{S}$, the following equation holds,

$$\frac{1}{2} \left(\frac{\partial}{\partial \alpha} + I \frac{\partial}{\partial \beta} \right) f_I(\alpha + I\beta) = 0.$$

Examples of slice regular functions are polynomials and power series of the form

$$\sum_{k=0}^{+\infty} q^k a_k, \quad \{a_k\}_{k \in \mathbb{N}} \subset \mathbb{H},$$

defined in their convergence set.

If Ω is a domain of \mathbb{H} such that $\Omega \cap \mathbb{R} = \emptyset$, then the previous definition, by itself, is not enough to obtain a satisfactory theory of regular functions. If, for instance $\Omega = \mathbb{H} \setminus \mathbb{R}$, then it is possible to construct the following example: consider the function $f : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$ defined as

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{H} \setminus \mathbb{C}_i \\ 0, & \text{if } x \in \mathbb{C}_i \setminus \mathbb{R}. \end{cases}$$

This function is of course regular but is not even continuous. So regularity, by itself, do not implies even continuity. However this example is quite meaningless since we could restrict to functions which are already differentiable. In section 3, we will show another example of differentiable regular function defined on $\mathbb{H} \setminus \mathbb{R}$ that has similar problems.

To solve this issue one can choose,

- (1) to study regular functions defined only over domains that do intersect the real axis;
- (2) to add some hypothesis to the set of functions.

Since we are interested, among the other things, in extending the theory for regular functions defined over any kind of domain, then we will use the second approach. More precisely we will use the concepts of *slice function* and of *stem function* introduced in [21] in a more general context. Using these instruments (that will be defined in the next section), it is possible to extend some rigidity and differential results, true for regular functions defined on domains which intersects the real axis (see [2, 3]).

After a brief summary of twistor theory of the 4-sphere and a review of the theory of slice regular functions, we will restore the theoretical work of [14] in our setting of slice regular functions on domains without real points. In particular, our point of view will be to describe the *twistor interpretation* of the theory of regular functions. With “twistor interpretation” we mean the correspondence given by the fact that any slice regular function $f : \Omega \rightarrow \mathbb{H}$ lifts to a (holomorphic) curve $\hat{f} : \mathcal{O} \subset \mathbb{CP}^3 \rightarrow \mathbb{CP}^3$, in the space \mathbb{CP}^3 (see section 4). The complex projective space \mathbb{CP}^3 is in fact the *twistor space* of $(\mathbb{S}^4 \simeq \mathbb{H} \cup \{\infty\}, g_{rnd})$, that is the total space of a bundle parameterizing orthogonal almost complex structures on \mathbb{S}^4 and we let $\pi : \mathbb{CP}^3 \rightarrow \mathbb{S}^4$ denotes the twistor projection with fibre \mathbb{CP}^1 . It is a well known fact (see, for instance, [26]), that complex hypersurfaces in \mathbb{CP}^3 produce OCSes on subdomains of \mathbb{S}^4 wherever such a hypersurface is a single-valued graph with

respect to the twistor projection and that any OCS J on a domain Ω generates a holomorphic hypersurface in \mathbb{CP}^3 .

With this in mind, instead of giving examples of OCSeS defined on some particular domain, we will give classes of surfaces in the twistor space of \mathbb{S}^4 that can be described with slice regular functions (i.e.: that can be interpreted as the image of the lift of a slice regular function). One of the main result is that not all the surfaces fit in this construction: first of all, they have to be ruled by lines. Thanks to this peculiarity we found interesting to explore a little bit more the so-called *twistor transform*, which, given a slice regular function, returns the disposition of lines whose lift carries on. This construction is formalized in view of the work [28], where many properties of submanifolds in the twistor space are studied from the arrangement of lines lying on them.

At the end we will study a very particular case that fit very well in our theory.

Finally, we briefly describe the structure of the present paper. In section 2 we summarize the main results in the twistor theory of \mathbb{S}^4 . In section 3 we present in a concise way the theory of slice regular functions. The only original part of this section is the one regarding *slice affine functions*. Then, in section 4, we effectively restore, in our more general context, the theory of [14], prove the main theorems and analyze classes of surfaces up to degree 3 in \mathbb{CP}^3 that can be reached by the lift of a slice regular function. In section 5 we show that (almost) any rational curve in the Grassmannian $\mathbb{Gr}_2(\mathbb{C}^4)$ of 2-planes in \mathbb{C}^4 (interpreted as Plücker quadric in \mathbb{CP}^5), can be seen as the twistor transform of a slice regular function. Then, we exhibit the set of slice regular functions whose twistor transform describe a rational line inside $\mathbb{Gr}_2(\mathbb{C}^4)$. This result shows, in particular, the role of slice regular function not defined on \mathbb{R} . Indeed in the last remark of the section it is pointed out that, this set, doesn't contain any slice regular function defined over the reals.

The subsequent section contains an explicit example of application. This example, provides some techniques that will be surely involved in the future, in the study of much more complicated examples. In some sense these explicit computations are natural because they regard the study of a particular degree one surface (that is a hyperplane), in \mathbb{CP}^3 , hence a *basic case* of study.

At the end there is a final small section on the possible future developments of the present work.

2. TWISTOR SPACE OF \mathbb{S}^4

In this section we will review some aspect of twistor geometry focusing on the special case of \mathbb{S}^4 . This part of the paper does not contain any new result but is intended to be a summary of the main concepts and constructions that justify our study. The themes that we are going to describe are classical but, according to our notation and language, we refer to the following more recent papers [4, 5, 26, 28].

The twistor space Z of an oriented Riemannian manifold (M, g) is the total space of a bundle containing almost complex structures (ACS) defined on M compatible with the metric g and the orientation. The definition of the twistor space does not depend on the full metric g but only on its conformal class $[g]$. In fact, if J is an ACS on M compatible with g and $g' = e^f g \in [g]$, then J is obviously compatible with respect to g' as well.

A motivation to study twistor spaces is that, if M is half-conformally-flat¹, then its conformal geometry is encoded into the complex geometry of Z : for instance an ACS on M compatible with the metric is integrable if and only if the corresponding section of Z defines a holomorphic submanifold.

Of course we will focus in the case in which M is the 4-sphere $\mathbb{S}^4 \simeq \mathbb{HP}^1$, which topologically is $\mathbb{R}^4 \cup \{\infty\}$. Here the quaternionic projective line \mathbb{HP}^1 is defined to be the set of equivalence classes $[q_1, q_2]$, where $[q_1, q_2] = [pq_1, pq_2]$ for any $p \in \mathbb{H} \setminus \{0\}$. As we will see, the choice of left multiplication is forced by the choice of study left slice functions. Moreover, we embed the quaternionic space \mathbb{H} into \mathbb{HP}^1 as $q \mapsto [1, q]$. So, the point at infinity is represented by $[0, 1]$. In this case, the twistor space is \mathbb{CP}^3 and the associated bundle structure $\pi : \mathbb{CP}^3 \rightarrow \mathbb{HP}^1$ is the fibration, defined as:

$$\pi[X_0, X_1, X_2, X_3] = [X_0 + X_1j, X_2 + X_3j].$$

¹Recall that a Riemannian metric g on M is called *half-conformally-flat*, or *anti-self-dual*, if the self-dual part W_+ of the Weyl tensor vanishes. Vanishing of both self-dual and anti-self-dual part of the Weyl tensor (i.e.: vanishing of the entire Weyl tensor), is equivalent to local conformal flatness of the metric g .

It is known (see e.g. [26], Section 2.6), that complex hypersurfaces in \mathbb{CP}^3 transverse to the fibres of π produce OCSes on subdomains of \mathbb{R}^4 whenever such a hypersurface is a single valued graph (with respect to the twistor projection). Vice versa, any OCS on a domain $\Omega \subset \mathbb{S}^4$ corresponds to a holomorphic hypersurface. Moreover, for topological reason it is not possible to define any ACS on the whole \mathbb{S}^4 (see Proposition 6.6 in [31]), so, no hypersurface in \mathbb{CP}^3 can intersect every fibre of the twistor fibration in exactly one point.

Remark 1. With our identifications, under the projection π , the matrix J of the ACS corresponding to the point $[1, u = x + iy, X_2, X_3] \in \mathbb{CP}^3$ is given by (up to notation and chirality, see Section 2 of [26]):

$$J = \frac{-1}{1 + |u|^2} \begin{pmatrix} 0 & 1 - |u|^2 & 2y & -2x \\ -1 + |u|^2 & 0 & -2x & -2y \\ -2y & 2x & 0 & 1 - |u|^2 \\ 2x & 2y & 1 - |u|^2 & 0 \end{pmatrix}.$$

From the previous simple considerations it is natural to investigate the algebraic geometry of surfaces in \mathbb{CP}^3 from this perspective. For instance, a natural question that arises is to classify surfaces of degree d in complex projective space up to conformal transformations of the base space \mathbb{S}^4 . A starting point, in this framework, is to find *conformal invariants*, but first we need to clarify what we mean for *conformal transformation* in the twistor space \mathbb{CP}^3 of \mathbb{S}^4 .

On any twistor fibre one can define a map j which sends an ACS J to $-J$. In our case j is exactly the action of multiplying a 1-dimensional complex subspace of \mathbb{C}^4 by the quaternion j in order to get a new 1-dimensional space, i.e.: j is the map on \mathbb{CP}^3 induced by the quaternionic multiplication by j in \mathbb{HP}^1 :

$$j : [X_0, X_1, X_2, X_3] \mapsto [-\bar{X}_1, \bar{X}_0, -\bar{X}_3, \bar{X}_2].$$

The map j is an antiholomorphic involution of the twistor space to itself with no fixed points. Starting with such a map j , one can recover the twistor fibration: given a point $X \in \mathbb{CP}^3$ there is a unique projective line connecting X and $j(X)$ and these lines form the fibres. So, if a line l in \mathbb{CP}^3 is a fibre for π , then $l = j(l)$.

The conformal symmetries of \mathbb{S}^4 correspond to the group of invertible transformations

$$[q_1, q_2] \mapsto [q_1 d + q_2 c, q_1 b + q_2 a], \quad a, b, c, d \in \mathbb{H},$$

where the invertibility condition is given by the following equation,

$$|a|^2 |d|^2 + |b|^2 |c|^2 - 2 \operatorname{Re}(b^c d c^c a) \neq 0.$$

Restricting to the affine line $q \in \mathbb{H} \mapsto [1, q]$, the latter becomes the linear fractional transformation given by $q \mapsto (qc + d)^{-1}(qa + b)$. These transformations correspond to the projective transformations of \mathbb{CP}^3 that preserve j . We, therefore, say that two complex submanifolds of \mathbb{CP}^3 are *conformally equivalent* if they are projectively equivalent by a transformation that preserves j .

Definition 2. Let Σ be an algebraic hypersurface of degree d in \mathbb{CP}^3 . A *twistor fibre* (or *twistor line*) of Σ is a fibre of π which lies entirely within the surface Σ .

Moreover, we define the *discriminant locus* of Σ to be the set D of points $p \in D \subset \mathbb{S}^4$, such that $\pi^{-1}(p) \cap \Sigma$ has cardinality different from d .

The fibres of the twistor fibration are complex projective lines in \mathbb{CP}^3 : if, in fact, we fix a quaternion $q = q_1 + q_2 j$, then the fibre $\mathbb{CP}^1 = \pi^{-1}([1, q])$, is given by

$$[1, q] = \pi[X_0, X_1, X_2, X_3] = [X_0 + X_1 j, X_2 + X_3 j] = [1, (X_0 + X_1 j)^{-1}(X_2 + X_3 j)],$$

which translates into,

$$(X_0 + X_1 j)(q_1 + q_2 j) = X_2 + X_3 j \Leftrightarrow \begin{cases} X_2 = X_0 q_1 - X_1 \bar{q}_2 \\ X_3 = X_0 q_2 + X_1 \bar{q}_1 \end{cases}$$

The number of twistor fibres of an algebraic surface Σ is an invariant under conformal transformations. Of course, if the degree of Σ is d , then a generic fibre, intersecting Σ transversely, will

contain d points because the defining polynomial of the surface, when restricted to the fibre, gives a polynomial of degree d .

Example 1. The inversion $q \mapsto q^{-1}$ lifts to the automorphism of \mathbb{CP}^3 defined by:

$$[X_0, X_1, X_2, X_3] \mapsto [X_2, X_3, X_0, X_1].$$

We will now review some classes of algebraic manifolds in \mathbb{CP}^3 already studied from this point of view.

2.1. Lines. Consider two lines in \mathbb{CP}^3 . If both lines are fibres of π then they are conformal equivalent by an isometry of \mathbb{S}^4 sending the image of one line under π to the image of the other line. If a line is not a fibre of π then its image will be a round 2-sphere in \mathbb{S}^4 (corresponding to a 2-sphere or a 2-plane in \mathbb{R}^4). Given such a 2-sphere in \mathbb{S}^4 , there are two projective lines lying above it in \mathbb{CP}^3 : in this case if l is a line projecting on the 2-sphere, then the other line is $j(l)$ which, in this case, is disjoint from l (see Proposition 2.8 in [28]). Therefore, a line in \mathbb{CP}^3 is given by either an oriented 2-sphere or a point in \mathbb{S}^4 . Moreover, any two such 2-spheres are conformal equivalents (this geometric correspondence is explained in detail in [28]).

2.2. Planes. A plane in \mathbb{CP}^3 is given by a single linear equation of the form

$$c_0 X_0 + c_1 X_1 + c_2 X_2 + c_3 X_3 = 0,$$

where, for each $i = 1 \dots 4$, c_i are constant numbers. A plane in \mathbb{CP}^3 cannot be transverse to every fibre of π because it would then define a complex structure on the whole \mathbb{S}^4 and (as already said), this is not possible. Therefore, a plane, always contains at least one twistor fibre. Twistor fibres are always skew (otherwise they would project to the same point), while two lines in a plane always meet. Hence a plane always contains exactly one twistor fibre. If one picks another line in the plane transverse to the fibre, its image under π will be a 2-sphere. We can find a conformal transformation of \mathbb{S}^4 mapping any 2-sphere with a marked point to any other 2-sphere with a marked point (see again [28]). We deduce that any couple of planes in \mathbb{CP}^3 are conformally equivalents (and not only projectively).

2.3. Quadrics. Non-singular quadrics in \mathbb{CP}^3 can be classified under conformal transformations of the 4-sphere \mathbb{S}^4 .

Theorem 3 ([26]). *Any non-singular quadric hypersurface in \mathbb{CP}^3 is equivalent under the action of the conformal group of \mathbb{S}^4 to the zero set of*

$$(1) \quad e^{\lambda+i\nu} X_0^2 + e^{-\lambda+i\nu} X_1^2 + e^{\mu-i\nu} X_2^2 + e^{-\mu-i\nu} X_3^2,$$

or the zero set of

$$(2) \quad i(X_0^2 + X_1^2) + k(X_1 X_3 - X_0 X_2) + X_1 X_2 - X_0 X_3,$$

where in the first case a couple of parameters (λ, μ, ν) , (λ', μ', ν') define two quadrics in the same equivalence class if and only if (λ, μ, ν) and (λ', μ', ν') belong to the same orbit under the group Γ of transformation of \mathbb{R}^3 generated by the four maps

$$\begin{cases} (\lambda, \mu, \nu) \mapsto (\lambda, \mu, \nu + \frac{\pi}{2}) \\ (\lambda, \mu, \nu) \mapsto (-\lambda, \mu, \nu) \\ (\lambda, \mu, \nu) \mapsto (\lambda, -\mu, \nu) \\ (\lambda, \mu, \nu) \mapsto (\mu, \lambda, -\nu), \end{cases}$$

while $k \in [0, 1)$ is a complete invariant in the second case.

With this result the authors of [26] were able to describe the geometry of non-singular quadric surfaces under the twistor projection π .

Theorem 4 ([26]). *For any non-degenerate quadric \mathcal{Q} there are three possibilities.*

- (1) \mathcal{Q} is a real quadric with discriminant locus a circle in \mathbb{S}^4 and \mathcal{Q} contains all the twistor lines over the circle.

- (2) \mathcal{Q} contains exactly one or exactly two twistor lines. In these cases the discriminant locus is a singular torus pinched at one or two points, respectively.
- (3) \mathcal{Q} does not contain any twistor lines. In this case the discriminant locus is a torus $\mathbb{T}^2 \subset \mathbb{S}^4$ with a smooth unknotted embedding.

Moreover if \mathcal{Q} is the zero locus of the polynomial in (1) with $0 \leq \lambda \leq \mu$ and $0 \leq \nu < \pi/2$, then

- (1) \mathcal{Q} contains a family of twistor lines over a circle if and only if $\lambda = \mu = \nu = 0$,
- (2) \mathcal{Q} contains exactly two twistor lines if and only if $\lambda = \mu \neq 0$ and $\nu = \pi/2$,
- (3) \mathcal{Q} contains no twistor lines in the other cases.

Finally if \mathcal{Q} is the zero locus of the polynomial in 2 with $k \in [0, 1)$, then the corresponding quadric \mathcal{Q} contains exactly one twistor line.

Singular quadric surfaces are still not studied.

2.4. Cubics. In [4] it is proven that a non-singular cubic contains at most 5 twistor lines. Moreover for a generic set of 5 points lying on a 2-sphere in \mathbb{S}^4 there exists a one parameter family of projectively isomorphic but conformal non-isomorphic non-singular cubic surfaces with 5 twistor lines corresponding to the 5 points. The following result was proven in [4] where the authors begin the study of this topic for non-singular cubic surfaces.

Theorem 5 ([4]). *Given 5 points on a 2-sphere in \mathbb{S}^4 , there is a non-singular cubic surface with 5 twistor lines corresponding to these points if and only if no 4 of the points lie on a circle.*

2.5. Quartics. The only known result for quartic surfaces in this direction (in our knowledge), regards a singular quartic scrolls studied in section 7 of [14]. The quartic scroll \mathcal{K} is defined by the following equation

$$(X_1X_2 - X_0X_3)^2 + 2X_1X_0(X_1X_2 + X_0X_3) = 0.$$

Define now γ to be the parabola defined by

$$\gamma := \{t^2 + ti \mid t \in \mathbb{R}\} \subset \mathbb{C}_i,$$

and Γ the following paraboloid of revolution:

$$\Gamma := \{q_0 + jq_2 + kq_3 \mid q_0, q_2, q_3 \in \mathbb{R}, q_0 = \frac{1}{4} - (q_2^2 + q_3^2)\}.$$

The following theorem is the mentioned result.

Theorem 6 ([14]). *Let $q \in \mathbb{H}$. The cardinality of the fibre $\pi^{-1}(q) \cap \mathcal{K}$ is different from 4 in the following cases:*

- (1) $q \in \gamma$ if and only if $\pi^{-1}(q) \subset \mathcal{K}$;
- (2) $q \in \mathbb{C}_i \setminus \gamma$ if and only if $\pi^{-1}(q)$ contains exactly two singular points of \mathcal{K} ;
- (3) $q \in \Gamma \setminus \{\frac{1}{4}\}$ if and only if $\pi^{-1}(q)$ is tangent to \mathcal{K} at two smooth points.

This last case was “reverse engineered” from the study of the possible (non-constant) OCSes defined on $\mathbb{R}^4 \setminus \lambda$. As already said, we are interested in another point of view so, later we will change this point of view concentrating our attention in the relation between slice regular functions and twistor geometry starting from the geometry of surfaces in \mathbb{CP}^3 .

3. SLICE REGULAR FUNCTIONS

Let \mathbb{H} be the real algebra of quaternions. In \mathbb{H} we define the following subsets: the *sphere of imaginary units*,

$$\mathbb{S} := \{q \in \mathbb{H} \mid q^2 = -1\},$$

the sphere centered in a point $x = \alpha + I\beta \in \mathbb{H} \setminus \mathbb{R}$,

$$\mathbb{S}_x := \{y \in \mathbb{H} \mid y = \alpha + J\beta, J \in \mathbb{S}\},$$

and the family of *slices*, parametrized by $I \in \mathbb{S}$,

$$\mathbb{C}_I := \{x \in \mathbb{H} \mid x = \alpha + I\beta, \alpha, \beta \in \mathbb{R}\}.$$

Sometimes we will use the term *semislice* to indicate any closed set of the form $\mathbb{C}_I^+ := \{x \in \mathbb{H} \mid x = \alpha + I\beta, \alpha \in \mathbb{R}, \beta \geq 0\}$, for some fixed $I \in \mathbb{S}$. If $x = \alpha + I\beta$ is a quaternion, its *usual conjugation* will be denoted by $x^c = \alpha - I\beta$.

Given a domain $\Omega \subset \mathbb{H}$, a function $f : \Omega \rightarrow \mathbb{H}$ is said to be *Cullen regular* if, for any $I \in \mathbb{S}$, the restriction $f|_{\Omega \cap \mathbb{C}_I}$ is a holomorphic function with respect to I . In other words if, for any $I \in \mathbb{S}$ the following equation holds

$$(3) \quad \frac{1}{2} \left(\frac{\partial}{\partial \alpha} + I \frac{\partial}{\partial \beta} \right) f|_{\Omega \cap \mathbb{C}_I} \equiv 0.$$

The theory of Cullen regular functions was born to contain polynomials and power series of the form $\sum_{k \in \mathbb{N}} q^k a_k$, where $\{a_k\}_{k \in \mathbb{N}} \subset \mathbb{H}$. This theory, introduced by G.Gentili and D.Struppa in [19] and based on a definition by C.Cullen (see [8]), is revealing, in the last years, to be very rich and interesting both from a theoretical point of view and (as this paper and [14] show), from some applicative results.

Even though at first there was an explosion of results regarding, for instance, the rigid behavior of such regular functions [7, 9, 10, 13, 16, 17], and the possibility of expanding them in certain power series [15, 29, 30], the formalism used to introduce the theory results to be inadequate to study such functions defined over particular domains: namely, domains with empty intersection with the real axis. The most simple example of what could go wrong is the function $f : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$ defined to be equal to some constant $q_0 \in \mathbb{H}$ everywhere but on a fixed slice \mathbb{C}_J on which is equal to some different constant $q_1 \in \mathbb{H}$ (compare with the function defined in the introduction). Such functions of course satisfy equation 3, if restricted to any slice, but are not even continuous. Later in this section, when the key features of slice regularity will be outlined, we will show another most significative example of a class \mathcal{C}^∞ function which satisfies the definition of Cullen regularity but, for some reason, we don't want in our theory (see example 2). Therefore, to better study the theory of slice regular functions on general domains we need another approach. The approach that we will use is the one introduced by R.Ghiloni and A.Perotti in [21], which exploit the use of the so-called *stem functions* to define the class of continuous functions on which we will apply the definition of regularity. The use of stem functions might seems unnecessarily technical, however, using precisely these techniques many results in the theory of slice regular functions were extended to the more general case and many others were proven for the first time [1, 2, 3, 21, 22, 23]. We will, then, describe our set of functions, directly by using this approach.

Let $\mathbb{H}_{\mathbb{C}}$ denotes the real tensor product $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. An element of $\mathbb{H}_{\mathbb{C}}$ will be of the form $p = x + \sqrt{-1}y$, where x and y are quaternions. Given another element $q = z + \sqrt{-1}t$ in $\mathbb{H}_{\mathbb{C}}$, we define the following product,

$$pq = xz - yt + \sqrt{-1}(xt + yz).$$

Of course $\sqrt{-1}$ plays the role of a complex structure in $\mathbb{H}_{\mathbb{C}}$. With the previous product, the space $\mathbb{H}_{\mathbb{C}}$ results to be a complex alternative algebra with unity. Given an element $p = x + \sqrt{-1}y \in \mathbb{H}_{\mathbb{C}}$ we define the following two commuting conjugations:

- $p^c = x^c + \sqrt{-1}y^c$;
- $\bar{p} = \bar{x} + \sqrt{-1}\bar{y}$.

Definition 3 (Stem function). Given a domain D in \mathbb{C} a function $F : D \rightarrow \mathbb{H}_{\mathbb{C}}$ is said to be a *stem function* if, for any $z \in D$ such that $\bar{z} \in D$ one has that $F(\bar{z}) = \overline{F(z)}$.

The condition in previous definition translates in the following way: a function F from a domain $D \subset \mathbb{C}$ to $\mathbb{H}_{\mathbb{C}}$ represents as $F(\alpha + i\beta) = F_1(\alpha + i\beta) + \sqrt{-1}F_2(\alpha + i\beta)$, with F_1 and F_2 quaternions; then F is a stem function if F_1 and F_2 are even and odd with respect to β , respectively. For this reason there is no loss of generality in taking D to be symmetric with respect to the real axis.

We will say that a stem function $F = F_1 + \sqrt{-1}F_2$ has a certain regularity (e.g.: is of class \mathcal{C}^n , for some $n \in \mathbb{N} \cup \{\infty, \omega\}$), if its components have that regularity.

Definition 4 (Circularization). Given a set $D \subset \mathbb{C}$ we define its *circularization* as the subset Ω_D of \mathbb{H} determined by the following equality

$$\Omega_D := \{\alpha + I\beta \in \mathbb{H} \mid \alpha + i\beta \in D, I \in \mathbb{S}\}.$$

Sets of this form will be called *circular set*. For any $I \in \mathbb{S}$ we will use the following notation: $D_I = D \cap \mathbb{C}_I$.

Remark 2. If D is symmetric with respect to the real axis and $D \cap \mathbb{R} = \emptyset$, then $\Omega_D \simeq D^+ \times \mathbb{S}$, where D^+ is the intersection between D and the complex upper half plane: $D^+ = D \cap \mathbb{C}^+$.

Definition 5 (Slice function). Let Ω_D be a circular set in \mathbb{H} . A function $f : \Omega_D \rightarrow \mathbb{H}$ is said to be a (left) *slice function* if it is induced by a stem function $F = F_1 + \sqrt{-1}F_2$ (denoted by $f = \mathcal{I}(F)$), in the following way:

$$f(\alpha + I\beta) = F_1(\alpha + i\beta) + IF_2(\alpha + i\beta), \quad \forall \alpha + I\beta \in \Omega_D.$$

The family of slice functions defined over some circular set Ω_D will be denoted by $\mathcal{S}(\Omega_D)$. We will say that a slice function $f = \mathcal{I}(F)$ has a certain regularity (e.g.: is of class \mathcal{C}^n , for some $n \in \mathbb{N} \cup \{\infty, \omega\}$) if the inducing stem function F has that regularity. The space of slice functions of class \mathcal{C}^n defined over Ω_D will be denoted by $\mathcal{S}^n(\Omega_D)$.

For each n , the family $\mathcal{S}^n(\Omega_D)$ is a real vector space and quaternionic right module, i.e.: for any $f, g \in \mathcal{S}^n(\Omega_D)$, for any $c \in \mathbb{R}$ and for any $q \in \mathbb{H}$, $cf + gq \in \mathcal{S}^n(\Omega_D)$.

Of course one can define analogously *right* slice function, by putting, in the previous definition, the complex imaginary unit at the right of F_2 . The uprising theory would be completely symmetric with the one described here.

The nature of stem functions yields the well-posedness of the slice function's definition. In fact if $f = \mathcal{I}(F_1 + \sqrt{-1}F_2) : \Omega_D \rightarrow \mathbb{H}$ is a slice function and $x = \alpha + I\beta$ is a quaternion in Ω_D , then $f(\alpha + (-I)(-\beta)) = F_1(\alpha - i\beta) - IF_2(\alpha - i\beta) = F_1(\alpha + i\beta) + IF_2(\alpha + i\beta) = f(\alpha + I\beta)$. Moreover one can see, from the definition, that a (left) slice function is nothing but a quaternionic function of one quaternionic variable that is *quaternionic (left) affine with respect to the imaginary unit*. This fact, together with the next (basic but) fundamental result, shows that there is not loss of generality in choosing circular sets as domains of definition for slice functions and that, for any slice function there is unique inducing stem function.

Theorem 7 (Representation formula, [21], Proposition 6). *Let $f \in \mathcal{S}(\Omega_D)$ be a slice function defined over any circular set Ω_D . Then, for any $J \neq K \in \mathbb{S}$, f is uniquely determined by its values over \mathbb{C}_J^+ and \mathbb{C}_K^+ by the following formula:*

$$f(\alpha + I\beta) = (I - K)(J - K)^{-1}f(\alpha + J\beta) - (I - J)(J - K)^{-1}f(\alpha + K\beta), \quad \forall \alpha + I\beta \in \Omega_D.$$

In particular if $K = -J$, we get the following simpler formula

$$(4) \quad f(x) = \frac{1}{2} [f(\alpha + J\beta) + f(\alpha - J\beta) - IJ(f(\alpha + J\beta) - f(\alpha - J\beta))].$$

This well-known result can be easily proven having in mind that a straight line parametrized by an affine function in an affine space can be recovered simply by two of its values. From the representation formula it is easy to see that any slice function sends each sphere \mathbb{S}_x contained in its domain into another sphere \mathbb{S}_y for some $y \in \mathbb{H}$.

Given any slice function, it is possible to define its spherical derivative as follows.

Definition 6 (Spherical derivative). Let $f = \mathcal{I}(F_1 + \sqrt{-1}F_2) \in \mathcal{S}(\Omega_D)$ and $x = \alpha + I\beta \in \Omega_D \setminus \mathbb{R}$. The *spherical derivative* of f is the slice function $\partial_s f$ induced by the stem function $F_2(z)/Im(z)$.

For any point $x \in \Omega_D \setminus \mathbb{R}$, the spherical derivative of a slice function f can be also defined as

$$\partial_s f(x) = \frac{1}{2} Im(x)^{-1} (f(x) - f(x^c)).$$

The spherical derivative of any slice function is constant on each sphere \mathbb{S}_x contained in the domain of definition of the function. Moreover, if the function f is of class at least \mathcal{C}^1 , then its spherical derivative can be extended continuously to the real line (see Proposition 7 in [21]).

As the reader can see, the spherical derivative of a slice function is not a genuine derivative, i.e.: it is not defined as some sort of limit of incremental ratio. However, as we will see in Theorem 19, it is the right quantity to control the behavior of a slice function alongside the spheres \mathbb{S}_x contained in its domain. In this view, we are going to define now the partial derivatives along

the remaining directions, which are the slices. First observe that, for a sufficiently regular stem function $F = F_1 + \sqrt{-1}F_2$, if $z = \alpha + i\beta$ is a point in the domain of F , the partial derivatives $\partial F_1/\partial\alpha$ and $\sqrt{-1}(\partial F_2/\partial\beta)$ are stem functions too.

Definition 7 (Slice Derivative). Given a function $f \in \mathcal{S}^1(\Omega_D)$ we define its *slice derivatives* as the following continue slice functions defined over Ω_D :

$$\begin{aligned}\frac{\partial f}{\partial x} &:= \mathcal{I} \left(\frac{\partial F}{\partial z} = \frac{1}{2} \left(\frac{\partial F}{\partial \alpha} - \sqrt{-1} \frac{\partial F}{\partial \beta} \right) \right) \\ \frac{\partial f}{\partial x^c} &:= \mathcal{I} \left(\frac{\partial F}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial F}{\partial \alpha} + \sqrt{-1} \frac{\partial F}{\partial \beta} \right) \right)\end{aligned}$$

As we said before, $\sqrt{-1}$ is a complex structure for $\mathbb{H}_{\mathbb{C}}$. But then a stem function $F : D \rightarrow \mathbb{H}_{\mathbb{C}}$ is holomorphic if $\partial F/\partial \bar{z} = 0$. In this way we naturally define slice regularity as follows.

Definition 8 (Slice regularity). A (left) slice function $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D)$ is said to be *(left) slice regular* if the slice derivative $\partial f/\partial x^c$ vanishes everywhere. The set of slice regular functions defined over a certain domain Ω_D will be denoted by $\mathcal{SR}(\Omega_D)$.

The set $\mathcal{SR}(\Omega_D)$ is a real vector space and quaternionic right module. Again, the theory does not change if we consider right slice regular function instead of left ones.

Remark 3. Due to the representation formula, a slice function f is slice regular if and only if its restriction to any slice \mathbb{C}_I is a holomorphic function with respect to the complex imaginary unit defined both on the domain and image as left quaternionic multiplication by I . This is true if and only if for each couple of imaginary units $I \neq J \in \mathbb{S}$ the restrictions of f on the semislices defined by I and J are holomorphic functions with respect to I and J , respectively. Another fact about slice regular functions is that, in analogy with the holomorphic framework, if a function f is slice regular, then its slice derivative $\partial f/\partial x$ is regular as well. All these facts can be found in [21].

There is a formula that links the value of the spherical and the slice derivatives and it is exposed in the next proposition.

Proposition 8 ([3], Proposition 12). *Let $f \in \mathcal{SR}(\Omega_D)$ be a slice regular function, then the following formula holds:*

$$\frac{\partial f}{\partial x}(x) = 2\text{Im}(x) \left(\frac{\partial}{\partial x} \partial_s f \right)(x) + \partial_s f(x), \quad \forall x = \alpha + J\beta \in \Omega_D.$$

Any slice regular function is Cullen regular (see [18], Definition 1.1), but, if the domain of definition does not intersects the real line then the converse is not true in general. This issue was studied in [22], where the authors show that asking for a generic quaternionic function defined over a domain without real points to be Cullen regular is not enough to obtain a satisfactory theory: in general you lost *sliceness* that is equivalent to the representation formula, so many fundamental theorems lost any chances to be true in this very general context. However the author of the present paper believes that this issue should be further studied: some interesting subclasses might arise from this study.

Example 2. This example shows a \mathcal{C}^∞ quaternionic functions of one quaternionic variable that is Cullen regular but not slice regular. Fix $J \in \mathbb{S}$ and a real number $\lambda \notin \{-1, 0, 1\}$. Let $x = \alpha + I_x\beta$ be a non-real quaternion and define $f : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$ as

$$f(x) = I_x + \lambda J I_x J.$$

This function, which is of class \mathcal{C}^∞ and Cullen regular, sends $\mathbb{H} \setminus \mathbb{R}$ into an ellipsoidal surface. Since any slice function sends 2-spheres of the form \mathbb{S}_x to other 2-spheres of the same form, this implies that f is not a slice function.

A consolidated and well known result about slice regular functions is the *splitting lemma*. It says that any slice regular function, if properly restricted, admits a splitting into two actual complex holomorphic functions. A proof of this result can be found in [7, 23], the first with the additional hypothesis that the domain of definition intersects the real axis.

Lemma 9 ([7, 23]). *Let $f \in \mathcal{SR}(\Omega_D)$. Then, for each $J \in \mathbb{S}$ and each $K \perp J$, $K \in \mathbb{S}$, there exist two holomorphic functions $G, H : D_J \rightarrow \mathbb{C}_J$ such that*

$$f_J = G + HK.$$

Observe that G and H are defined over the whole D_J . This means that, if D_J is disconnected and the disjoint union of D_1 and D_2 , then, G and H could have unrelated different behavior on D_1 and D_2 . A particular case is when $\Omega_D \cap \mathbb{R} = \emptyset$, where, *a priori* the function G and H can have different behaviors if restricted either to D_J^+ or D_J^- .

Speaking now about operations between slice functions, in general, their pointwise product is not a slice function². However, there exists another notion of product which works well in our context. The following, introduced in [7, 17] for slice regular functions defined over domains that does intersect \mathbb{R} and in Definition 9 of [21] for slice functions (in the context of real alternative algebras), is the notion that we will use.

Definition 9 (Slice product). Let $f = \mathcal{I}(F)$, $g = \mathcal{I}(G) \in \mathcal{S}(\Omega_D)$ the (slice) product of f and g is the slice function

$$f \cdot g := \mathcal{I}(FG) \in \mathcal{S}(\Omega_D).$$

Explicitly, if $F = F_1 + \sqrt{-1}F_2$ and $G = G_1 + \sqrt{-1}G_2$ are stem functions, then $FG = F_1G_1 - F_2G_2 + \sqrt{-1}(F_1G_2 + F_2G_1)$.

Remark 4. Let $f(x) = \sum_j x^j a_j$ and $g(x) = \sum_k x^k b_k$ be polynomials or, more generally, converging power series with coefficients $a_j, b_k \in \mathbb{H}$. The usual product of polynomials, where x is considered to be a commuting variable, can be extended to power series in the following way: the star product $f * g$ of f and g is the convergent power series defined by setting

$$(f * g)(x) := \sum_n x^n \left(\sum_{j+k=n} a_j b_k \right).$$

In Proposition 12 of [21] it was proven that the product of f and g , viewed as slice functions, coincide with the star product $f * g$, i.e.: $\mathcal{I}(FG) = \mathcal{I}(F) * \mathcal{I}(G)$. Indeed sometimes the slice product between f and g is denoted by $f * g$ (see [16] or [19]) and called *regular product*, to stress the fact that this notion of product was born to preserve the regularity. The next proposition precise this fact.

Proposition 10 ([21], Proposition 11). *If $f, g \in \mathcal{SR}(\Omega_D)$ then $f \cdot g \in \mathcal{SR}(\Omega_D)$*

In [21] it is also pointed out and proved that the regular product introduced in [7, 17] is generalized by this one if the domain Ω_D does not have real points. By the way, an idea to prove this theorem is simply to explicit the slice product in term of stem function and compute the Cauchy-Riemann equations.

The slice product of two slice functions coincide with the punctual product if the first slice function is *real* (see Definition 10 of [21]).

Definition 10 (Real slice function). A slice function $f = \mathcal{I}(F)$ is called *real* or *slice-preserving* or, even, *quaternionic intrinsec* if the \mathbb{H} -valued components F_1, F_2 are real valued.

The next proposition, stated in Lemma 6.8 of [20], justifies the different names given in the previous definition.

Proposition 11. *Let $f = \mathcal{I}(F)$ be a slice function. The following conditions are equivalent.*

- *f is real.*
- *For all $J \in \mathbb{S}$, $f(D_J) \subset \mathbb{C}_J$.*
- *For all x in the domain of f it holds $f(x) = (f(x^c))^c$.*

²For instance, if $f(q) = qa$ and $g(q) = q$, with $a \in \mathbb{H} \setminus \mathbb{R}$, then $h(q) = f(q)g(q) = qaq$ is not a slice function.

These functions are special since, in a certain sense, transpose the concept of complex function in our setting. In fact, if $h(z) = u(z) + iv(z)$ is a complex function defined over a certain domain $D \subset \mathbb{C}$ with $D \cap \mathbb{R} \neq \emptyset$, then the function $H : D \rightarrow \mathbb{H}_{\mathbb{C}}$ defined as $H(z) = u(z) + \sqrt{-1}v(z)$ is a stem function, and $\mathcal{I}(H)$ is a real slice function.

As stated in [17], if f is a slice regular function defined on $B(0, R)$, the ball of center zero and radius R for some $R > 0$, then it is real if and only if f can be expressed as a power series of the form

$$f(x) = \sum_{n \in \mathbb{N}} x^n a_n,$$

with a_n real numbers.

By a simple computation, it is possible to prove the following lemma.

Lemma 12 ([2], Lemma 2.12). *Let $f = \mathcal{I}(F), g = \mathcal{I}(G) \in \mathcal{S}(\Omega_D)$, with f real, then the slice function $h : \Omega_D \rightarrow \mathbb{H}$, defined by $h := f \cdot g$ is such that*

$$h(x) = f(x)g(x).$$

Now, we are going to define an “inversion” for slice functions. The following first two definitions appeared for the first time in [7], can be found also in [16] and [17]. Later they were generalized by Ghiloni and Perotti for slice functions in Definition 11 of [21]. The definition of slice reciprocal was firstly introduced in [7, 17, 16, 30] and then in [2] if the domain of definition has empty intersection with the real line. Let us denote the zero set of a slice function f by $V(f)$.

Definition 11 (Slice conjugate, Normal function, Slice reciprocal). Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$, then also $F(z)^c := F_1(z)^c + \sqrt{-1}F_2(z)^c$ is a stem function. We set

- $f^c := \mathcal{I}(F^c) \in \mathcal{S}(\Omega_D)$, called *slice conjugate* of f ;
- $N(f) := f^c \cdot f$ *symmetrization* or *normal function* of f (the symmetrization of f is sometimes denoted by f^s).

Let $f = \mathcal{I}(F) \in \mathcal{SR}(\Omega_D)$. We call the *slice reciprocal* of f the slice function

$$f^{-\cdot} : \Omega_D \setminus V(N(f)) \rightarrow \mathbb{H}$$

defined by

$$f^{-\cdot} = \mathcal{I}((F^c F)^{-1} F^c)$$

From the previous definition it follows that, if $x \in \Omega_D \setminus V(N(f))$, then

$$f^{-\cdot}(x) = (N(f)(x))^{-1} f^c(x).$$

Various results about the previous functions were reviewed in [1], including the fact that if a function f is slice regular then f^c , $N(f)$ and $f^{-\cdot}$ (where it is defined) are all regular. The notion of slice reciprocal was engineered such that, if f is a slice regular function with empty zero-locus then,

$$f \cdot f^{-\cdot} = f^{-\cdot} \cdot f = 1.$$

For more information about this result (including its proof) see, again, [1]. We are going now to review the construction of slice forms introduced by the author of the present paper in [1, 2]. We will start with the following general definition.

Definition 12. Let $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D)$. We define the *slice differential* $d_{sl}f$ of f as the following differential form:

$$\begin{aligned} d_{sl}f : (\Omega_D \setminus \mathbb{R}) &\rightarrow \mathbb{H}^*, \\ \alpha + I\beta &\mapsto dF_1(\alpha + i\beta) + IdF_2(\alpha + i\beta). \end{aligned}$$

Remark 5. The one-form $\omega : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}^*$ defined as $\omega(\alpha + I\beta) = Id\beta$, represents the outer radial direction to the sphere $\mathbb{S}_x = \{\alpha + K\beta \mid K \in \mathbb{S}\}$. Then $\omega(\alpha + I(-\beta)) = \omega(\alpha + (-I)\beta) = -\omega(\alpha + I\beta)$. We can translate this observation in the language of slice forms. The function $h(x) = Im(x)$ is a slice function induced by $H(z) = \sqrt{-1}Im(z)$. Then we have $d_{sl}h(\alpha + I\beta) = Id\beta(\alpha + i\beta)$ and, thanks to the previous considerations $d_{sl}h(\alpha + (-I)(-\beta)) = -Id\beta(\alpha - i\beta) = Id\beta(\alpha + i\beta)$. Summarizing, we have that $d\beta(\bar{z}) = -d\beta(z)$. The same doesn't hold for $d\alpha$ which is a constant

one form over \mathbb{H} and for this reason in the next computations we will omit the variable (i.e.: $d\alpha = d\alpha(z) = d\alpha(\bar{z})$).

In Proposition 10 of [3] it is proved that definition 12 is well posed, i.e. if D is symmetric with respect to the real axis, then

$$d_{sl}f(\alpha + I\beta) = d_{sl}f(\alpha + (-I)(-\beta)), \quad \forall \alpha + I\beta \in \Omega_D \setminus \mathbb{R}$$

We can represent, then, the slice differential as follows.

Proposition 13 ([3], Proposition 11). *Let $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D)$ with $D \subset \mathbb{C}^+$ (so that $\beta > 0$). Then, on $\Omega_D \setminus \mathbb{R}$, the following equality holds true.*

$$d_{sl}f = \frac{\partial f}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \beta} d\beta.$$

It is clear from the definition that, if we choose the usual coordinate system, where $x = \alpha + I\beta$ with $\beta > 0$, then $d_{sl}x = d\alpha + Id\beta$ and $d_{sl}x^c = d\alpha - Id\beta$. We can now state the following theorem.

Theorem 14 ([3], Theorem 7). *Let $f \in \mathcal{S}^1(\Omega_D)$. Then the following equality holds:*

$$d_{sl}x \frac{\partial f}{\partial x}(x) + d_{sl}x^c \frac{\partial f}{\partial x^c}(x) = d_{sl}f(x), \quad \forall x \in \Omega_D \setminus \mathbb{R}.$$

We have then the obvious corollary:

Corollary 15. *Let $f \in \mathcal{SR}(\Omega_D)$. Then the following equality holds:*

$$d_{sl}x \frac{\partial f}{\partial x}(x) = d_{sl}f(x), \quad \forall x \in \Omega_D \setminus \mathbb{R}.$$

Some important classes of slice regular functions are now introduced.

Definition 13 (Slice constant function). Let Ω_D be a connected circular domain and let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$. f is called *slice constant* if the stem function F is locally constant on D .

Proposition 16 ([2], Proposition 3.3 and Theorem 3.4). *Let $f \in \mathcal{S}(\Omega_D)$ be a slice function. If f is slice constant then it is slice regular. Moreover f is slice constant if and only if*

$$\frac{\partial f}{\partial x} \equiv 0.$$

Remark 6. The previous proposition tells that if we have a slice constant function $f \in \mathcal{SR}(\Omega_D)$ over a connected circular domain Ω_D , then, given $J \in \mathbb{S}$, if $x \in D_J^+ \setminus \mathbb{R}$

$$f(x) = a + Jb = a + \frac{Im(x)}{||Im(x)||} b, \quad a, b \in \mathbb{H}.$$

Proposition 17. *Let Ω_D be a connected circular domain. Let $g : \Omega_D \rightarrow \mathbb{H}$ be a slice function. g is slice constant if and only if given any fixed $J \in \mathbb{S}$, $g|_{\Omega_D \setminus \mathbb{R}}$ is a linear combination, with right quaternionic coefficients, of the two functions $g_+, g_- : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$ defined by $g_{\pm}(\alpha + I\beta) = 1 \pm IJ$.*

Proof. Thanks to Theorem 16, any linear combination of the two functions g_+, g_- is slice constant since their slice derivative is everywhere zero. Vice versa, given a slice constant function $g = \mathcal{I}(G) : \Omega_D \rightarrow \mathbb{H}$, with $G := g_1 + \sqrt{-1}g_2$, its locally constant stem function, it holds $g(\alpha + I\beta) = g_1 + Ig_2$, but, thanks to the representation formula 4, for any $J \in \mathbb{S}$ we have $g(\alpha + I\beta) = [(1 - IJ)(g_1 + Jg_2) + (1 + IJ)(g_1 - Jg_2)]/2$. \square

Now we will introduce the set of slice regular function that are affine slice by slice. This notion will be useful in a next result.

Definition 14 (Slice affine functions). Let $f : \Omega_D \rightarrow \mathbb{H}$ be a slice regular function. f is called *slice affine* if its slice derivative is a slice constant function.

Proposition 18. *Let $f : \Omega_D \rightarrow \mathbb{H}$ be a slice function. f is slice affine if and only if given any fixed $J \in \mathbb{S}$, $f|_{\Omega_D \setminus \mathbb{R}}$ is a linear combination, with right quaternionic coefficient, of the four functions $f_+, f_-, g_+, g_- : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$, where g_+, g_- are the same as before and $f_{\pm}(\alpha + I\beta) = (\alpha + I\beta)g_{\pm}(\alpha + I\beta)$.*

Proof. If f is a linear combination of f_+, f_- and g_+, g_- then it is obviously a slice affine function. Vice versa, since $\partial f / \partial x$ is a slice constant function, then, in the language of slice forms

$$d_{sl}f = d_{sl}x \frac{\partial f}{\partial x} = d_{sl}x g(x),$$

with $g = \mathcal{I}(g_1 + \sqrt{-1}g_2)$ a slice constant function. The previous equality, using the definition of slice form, is equivalent to the following one

$$\frac{\partial F_1}{\partial \alpha} d\alpha + \frac{\partial F_1}{\partial \beta} d\beta + I \left(\frac{\partial F_2}{\partial \alpha} d\alpha + \frac{\partial F_2}{\partial \beta} d\beta \right) = g_1 d\alpha - g_2 d\beta + I(g_2 d\alpha + g_1 d\beta),$$

that implies $F_1 = g_1\alpha - g_2\beta + q_1$ and $F_2 = g_2\alpha + g_1\beta + q_2$, for some couple q_1, q_2 of quaternions. But then, applying the representation formula in 4, and using the same argument as in the proof of the previous theorem we obtain,

$$\begin{aligned} f(\alpha + I\beta) &= g_1\alpha - g_2\beta + I(g_2\alpha + g_1\beta) + q_1 + Iq_2 \\ &= \alpha(g_1 + Ig_2) + I\beta(g_1 + Ig_2) + q_1 + Iq_2 \\ &= (\alpha + I\beta)(g_1 + Ig_2) + q_1 + Iq_2 \\ &= (\alpha + I\beta)[(1 - IJ)(g_1 + Jg_2) + (1 + IJ)(g_1 - Jg_2)]/2 + \\ &\quad + [(1 - IJ)(q_1 + Jq_2) + (1 + IJ)(q_1 - Jq_2)]/2. \end{aligned}$$

□

Remark 7. The set of slice constant functions contains the set of constant functions and the condition for a slice constant function $g = g_+q_+ + g_-q_-$ to be extended to \mathbb{R} is that $q_+ = q_-$ (i.e.: g is a constant function). Analogously, a slice affine function $f = f_+q_{1+} + f_-q_{1-} + g_+q_{0+} + g_-q_{0-}$ extends to the real line if and only if $q_{1+} = q_{1-}$ and $q_{0+} = q_{0-}$ (i.e.: $f = xa + b$ is a \mathbb{H} -affine function). For slice constant function the assertion is trivial while for slice affine functions it requires a simple consideration regarding the limit of the function for β that approach 0 when β is lower or greater than zero. In formula, the previous condition is the following one:

$$\lim_{\substack{\beta \rightarrow 0 \\ \alpha + I\beta \in \mathbb{C}_I^+}} f(\alpha + I\beta) = \lim_{\substack{\beta \rightarrow 0 \\ \alpha + I\beta \in \mathbb{C}_I^-}} f(\alpha + I\beta).$$

Remark 8. One can define, in general, the class of “slice polynomial” functions as the set of slice regular functions such that the n^{th} slice derivative vanishes for some n . This can be actually a useful notion in view of some researches regarding the number of counterimages of a slice regular function defined over a domain without real points. Anyway this theme is not explored in this paper and so we will not spend any other words.

In the next part of this section we will remember some theorems regarding the nature of the real differential of a slice regular function. These results can be found in [29, 14] and their generalization in [3]. Firstly we will expose a representation of the differential. As we said before, given a slice regular function, its spherical derivative and its slice derivative control the variation of the functions along spheres \mathbb{S}_x and slices \mathbb{C}_I , respectively. This remark is in fact the content of the following theorem.

Theorem 19 ([3, 29]). *Let $f \in \mathcal{SR}(\Omega_D)$ and let $(df)_x$ denote the real differential of f at $x = \alpha + I\beta \in \Omega_D \setminus \mathbb{R}$. If we identify $T_x\Omega_D$ with $\mathbb{H} = \mathbb{C}_I \oplus \mathbb{C}_I^\perp$, then for all $v_1 \in \mathbb{C}_I$ and $v_2 \in \mathbb{C}_I^\perp$,*

$$(df)_x(v_1 + v_2) = v_1 \frac{\partial f}{\partial x}(x) + v_2 \partial_s f(x).$$

If $\alpha \in \Omega_D \cap \mathbb{R}$ then, the previous formula becomes the following one

$$(df)_\alpha(v) = v \frac{\partial f}{\partial x}(\alpha) = v \partial_s f(\alpha).$$

Proposition 20 ([14, Proposition 3.3]). *Let $f \in \mathcal{SR}(\Omega_D)$ and $x_0 = \alpha + J\beta \in \Omega_D \setminus \mathbb{R}$.*

- *If $\partial_s f(x_0) = 0$ then:*
 - *df_{x_0} has rank 2 if $\frac{\partial f}{\partial x}(x_0) \neq 0$;*
 - *df_{x_0} has rank 0 if $\frac{\partial f}{\partial x}(x_0) = 0$.*

- If $\partial_s f(x_0) \neq 0$ then df_{x_0} is not invertible at x_0 if and only if $\frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1}$ belongs to \mathbb{C}_f^\perp .

Let now $\alpha \in \Omega_D \cap \mathbb{R}$. df_{x_0} is invertible at α if and only if its rank is not 0 at $x_0 = \alpha + J\beta$. This happens if and only if $\partial_s f(x_0) = \frac{\partial f}{\partial x}(x_0) \neq 0$.

Definition 15 (Singular set). Let $f : \Omega \rightarrow \mathbb{H}$ any quaternionic function of quaternionic variable. We define the *singular set* of f as

$$N_f := \{x \in \Omega \mid df \text{ is not invertible at } x\}.$$

Given a slice regular function f that is not slice constant, then its singular set N_f is closed with empty interior, moreover if f is injective then its spherical and slice derivatives are both nonzero. With some other information it is possible to prove the following theorem.

Theorem 21 ([14, 3]). *Let f be an injective slice regular function, then $N_f = \emptyset$.*

4. TWISTOR LIFT

Starting from Theorem 2 (or equivalently from Theorem 4, part (1)), we know that, up to sign, on $\mathbb{H} \setminus \mathbb{R}$ it is possible to define only one non-constant OCS. This OCS can be defined as follows.

Definition 16 (Slice complex structure). Let $p = \alpha + I_p \beta \in X = \mathbb{H} \setminus \mathbb{R}$ with $\beta > 0$, and we are identifying $T_p X \simeq \mathbb{H}$. We define the following OCS over X :

$$\mathbb{J}_p v = \frac{Im(p)}{||Im(p)||} v = I_p v,$$

where v is a tangent vector of X in p and $I_p v$ denotes the quaternionic multiplication between I_p and v .

Later we will describe the algebraic surface in \mathbb{CP}^3 arising from this OCS, but now let us pass again to quaternionic functions. The whole story of the relation between slice regular functions and twistor geometry starts thanks to theorems 19 and 21 that extend two results proved, respectively in [29] and [14].

Given an injective slice regular function $f : \Omega_D \rightarrow \mathbb{H}$ we define the pushforward of \mathbb{J} via f on $f(\Omega_D \setminus \mathbb{R})$ as:

$$\mathbb{J}^f := (df)\mathbb{J}(df)^{-1},$$

for any $v \in T_{f(p)} f(\Omega_D \setminus \mathbb{R}) \simeq \mathbb{H}$.

The following theorem explains the action of the push-forward of \mathbb{J} via a slice regular function.

Theorem 22. *Let $f : \Omega_D \rightarrow \mathbb{H}$ be an injective slice regular function and $p = \alpha + I_p \beta \in \Omega_D$. Then*

$$\mathbb{J}_{f(p)}^f v = \frac{Im(p)}{||Im(p)||} v = I_p v.$$

Moreover \mathbb{J}^f is an OCS on the image of f .

Proof. The theorem can be proved as in [14], but we will write again the proof using the representation in Theorem 19 of the real differential of a slice regular function. The thesis follows thanks to the next computations. Let v be a tangent vector to $f(\Omega_D \setminus \mathbb{R})$ in $f(x)$

$$\mathbb{J}_{f(x)}^f v = (df)_x \mathbb{J}_x (df)_{f(x)}^{-1} v = \circledast.$$

Putting $(df)_{f(x)}^{-1} v = w$ and denoting by w_\top and w_\perp , respectively, the tangential and orthogonal part of w with respect to \mathbb{C}_{I_x} , we obtain,

$$\begin{aligned} \circledast &= (df)_x \mathbb{J}_x w = (df)_x I_x w \\ &= I_x w_\top \frac{\partial f}{\partial x}(x) + I_x w_\perp \partial_s f(x) \\ &= I_x (df)_x w = I_x v. \end{aligned}$$

For the second part of the theorem we refer again in [14], anyway, it is enough to compute the quantity $g_{Euc}(\mathbb{J}X, \mathbb{J}Y)$, point by point. \square

At this point one could ask if it is possible to construct a twistor theory also for slice regular functions that do not extend to the real line. To be more clear, if $\Omega_D \cap \mathbb{R} = \emptyset$, then *is it possible to construct its twistor lift as explained in Theorem 5.3 of [14]*? Well, the answer is yes and it is explained in the next pages.

First of all we need to introduce coordinates for the sphere \mathbb{S} of imaginary units. For this purpose we will follow the construction in section 4 of [14]. For any complex number u we define the following quaternion $Q_u := 1 + uj$. Let, now, ϕ be the following application:

$$\begin{aligned} \phi : \mathbb{C} \times \mathbb{C}^+ &\rightarrow \mathbb{H} \\ (u, v) &\mapsto Q_u^{-1}vQ_u \end{aligned}$$

By direct computation it is clear that $\phi(u, \alpha + J\beta) \in \alpha + \mathbb{S}\beta$ and so, for any $J \in \mathbb{S}$, the number $\phi(u, J)$ belongs to \mathbb{S} as well. Fix now J to be equal to i , then, for each $q \in \mathbb{H} \setminus \mathbb{R}$, there exists a unique couple $(u, v) \in \mathbb{C} \times \mathbb{C}^+$ such that $q = \phi(u, v)$. In particular, if $q = \alpha + I\beta$, $\beta > 0$, then,

$$\begin{cases} v &= \alpha + i\beta \\ u &= -i\frac{b+ic}{1+a}, \end{cases}$$

where $I = ai + bj + ck$. And, finally, we obtain the following representation,

$$\alpha + I\beta = \phi(u, v) = Q_u^{-1}vQ_u = \alpha + Q_u^{-1}iQ_u\beta.$$

We embed now $\mathbb{H} \setminus \mathbb{R}$ in \mathbb{HP}^1 via the function $q \rightarrow [1, q]$. Such embedding can be viewed, also, in the following way:

$$\begin{aligned} [1, q] &= [1, Q_u^{-1}vQ_u] = [Q_u, vQ_u] \\ &= [1 + uj, v + vu] = \pi[1, u, v, uv], \end{aligned}$$

and so, we have obtained, as in [14], the following proposition.

Proposition 23. *The complex manifold $(\mathbb{H} \setminus \mathbb{R}, \mathbb{J})$ is biholomorphic to the open subset \mathcal{Q}^+ of the quadric*

$$(5) \quad \mathcal{Q} = \{[X_0, X_1, X_2, X_3] \in \mathbb{CP}^3 \mid X_0X_3 = X_1X_2\},$$

such that at least one of the following conditions is satisfied:

- $X_0 \neq 0$ and $X_2/X_0 \in \mathbb{C}^+$,
- $X_1 \neq 0$ and $X_3/X_1 \in \mathbb{C}^+$.

The quadric \mathcal{Q} is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$, while \mathcal{Q}^+ is biholomorphic to the product of a sphere time a open cap $\mathbb{CP}^1 \times \mathbb{C}^+$.

Now we have all the ingredients to state the following theorem which generalizes Theorem 5.3 of [14].

Theorem 24. *Let D be a domain of \mathbb{C} and $\Omega_D \subset \mathbb{H}$ its circularization. Let $f : \Omega_D \rightarrow \mathbb{H}$ be any slice function. Then f admits a twistor lift to $\mathcal{O} = \pi^{-1}(\Omega_D \setminus \mathbb{R}) \cap \mathcal{Q}^+$, i.e.: there exist a function $\tilde{f} : \mathcal{O} \rightarrow \mathbb{CP}^3$, such that $\pi \circ \tilde{f} = f \circ \pi$. Moreover f is slice regular if and only if \tilde{f} is a holomorphic map.*

As we said, this theorem was already proven in [14], when the domain D has nonempty intersection with the real line and the function f is regular. Our proof contemplate also the case in which f does not extends to the real line and it is not regular, so it is more general. To add this extension we will use the previous described formalism of stem functions to which we add this trivial lemma that is a consequence of Lemma 6.11 of [20].

Lemma 25. *Let $f = \mathcal{I}(F) : \Omega_D \rightarrow \mathbb{H}$ be a slice function induced by the stem function $F : D \rightarrow \mathbb{H}_{\mathbb{C}}$. Then, for each couple $I, J \in \mathbb{S}$ such that $I \perp J$, there exist two stem functions $F^\top, F^\perp : D \rightarrow \mathbb{C}_I \otimes_{\mathbb{R}} \mathbb{C}$, such that $f = f^\top + f^\perp J$ with $f^\top = \mathcal{I}(F^\top)$, while $f^\perp = \mathcal{I}(F^\perp)$.*

Now we pass to the proof of Theorem 24.

Proof. Since f is a slice function, then it is induced by a stem function $F : D \rightarrow \mathbb{H}_{\mathbb{C}}$ such that, for $q = \alpha + I\beta \in \Omega_D$,

$$f(q) = f(\alpha + I\beta) = f(\alpha + Q_u^{-1}iQ_u\beta) = F_1(\alpha + i\beta) + Q_u^{-1}iQ_uF_2(\alpha + i\beta).$$

Thanks to the previous lemma f can be written also as $f = f^\top + f^\perp j$, with $f^\top = \mathcal{I}(F^\top)$, $f^\perp = \mathcal{I}(F^\perp)$, $F^\top, F^\perp : D \rightarrow \mathbb{C}_i \otimes_{\mathbb{R}} \mathbb{C}$. Now, each stem function splits into two components, $F^\top = F_1^\top + \sqrt{-1}F_2^\top$ and $F^\perp = F_1^\perp + \sqrt{-1}F_2^\perp$, and we define, for $i \in \mathbb{S}$, $F_i^\top = p_i \circ F^\top$ and $F_i^\perp = p_i \circ F^\perp$, where p_i is the map that sends $\sqrt{-1}$ to i (e.g.: if $w = x + \sqrt{-1}y \in \mathbb{H}_{\mathbb{C}}$, then $p_i(w) = x + iy$). To resume we have the following diagram

$$\begin{array}{ccc} D & \xrightarrow{F^\top, F^\perp} & \mathbb{C}_i \otimes_{\mathbb{R}} \mathbb{C} \\ & \searrow F_i^\top, F_i^\perp & \downarrow p_i \\ & & \mathbb{C}_i \end{array}$$

Letting finally $q = \alpha + I\beta$ and $v = \alpha + i\beta$ and remembering that $Q_u = 1 + uj$, we can compute,

$$\begin{aligned} [1, f(q)] &= [1, f(Q_u^{-1}(\alpha + i\beta)Q_u)] \\ &= [1, f(\alpha + Q_u^{-1}iQ_u\beta)] \\ &= [1, F_1^\top(v) + Q_u^{-1}iQ_uF_2^\top(v) + F_1^\perp(v)j + Q_u^{-1}iQ_uF_2^\perp(v)j] \\ &= [Q_u, F_1^\top + ujF_1^\top + iF_2^\top + uijF_2^\top + F_1^\perp j + ujF_1^\perp j + iF_2^\perp j + uijF_2^\perp j] = \circledast, \end{aligned}$$

where in the last equality we have omitted the variable v . Now, for any $w \in \mathbb{C}_i$, we have that $jw = w^c j$ and $jwj = -w^c$ and so, identifying \mathbb{C}_i with \mathbb{C} ,

$$\begin{aligned} \circledast &= [Q_u, F_1^\top + uF_1^{\top c}j + iF_2^\top + uiF_2^{\top c}j + F_1^\perp j - uF_1^{\perp c} + iF_2^\perp j - uiF_2^{\perp c}] \\ &= [Q_u, F_1^\top + iF_2^\top + (F_1^\perp + iF_2^\perp)j + u((F_1^{\top c} + iF_2^{\top c})j - (F_1^{\perp c} + iF_2^{\perp c}))] \end{aligned}$$

We finally obtain the coordinates of the lift:

$$(6) \quad \tilde{f}[1, u, v, uv] = [1, u, p_i \circ F^\top(v) - u(p_i \circ F^{\perp c}(v)), p_i \circ F^\perp(v) + u(p_i \circ F^{\top c}(v))].$$

But now, remembering that F^\top, F^\perp are holomorphic stem functions, then, we have that f is slice regular if and only if \tilde{f} is a holomorphic map. \square

Remark 9. Starting with a regular slice function f , one can repeat the computations in the following way

$$\begin{aligned} [1, f(q)] &= [1, f(Q_u^{-1}(\alpha + i\beta)Q_u)] \\ &= [1, f(\alpha + Q_u^{-1}iQ_u\beta)] \\ &= [1, F_1(\alpha + i\beta) + Q_u^{-1}iQ_uF_2(\alpha + i\beta)] \\ &= [Q_u, Q_uF_1(\alpha + i\beta) + iQ_uF_2(\alpha + i\beta)] \\ &= [1 + uj, (1 + uj)F_1(\alpha + i\beta) + i(1 + uj)F_2(\alpha + i\beta)] \\ &= [1 + uj, f(\alpha + i\beta) + ujf(\alpha - i\beta)] \\ &= [1 + uj, f(v) + ujf(\bar{v})] = \circledast. \end{aligned}$$

At this point, using the splitting in Lemma 9, we can write $f_i(v) = G(v) + H(v)j$, where $G, H : D_i \rightarrow \mathbb{C}_i$ are holomorphic functions. Now, if $D_i \cap \mathbb{R} = \emptyset$, then the behaviors of G and H over D_i^+ and D_i^- are, in general, unrelated. We write then

$$G(v) := \begin{cases} g(v) & v \in D_i^+ \\ \hat{g}(\bar{v}) & v \in D_i^- \end{cases}, \quad H(v) := \begin{cases} h(v) & v \in D_i^+ \\ \hat{h}(\bar{v}) & v \in D_i^- \end{cases},$$

where g, \hat{g}, h, \hat{h} are holomorphic functions defined on D_i^+ . This is done because, in the lift, the variable v belongs to \mathbb{C}^+ and can be done because D is symmetric with respect to the real axis. Note that, since \hat{g} and \hat{h} are holomorphic functions, then the two functions $v \mapsto \overline{\hat{g}(\bar{v})}$, $v \mapsto \overline{\hat{h}(\bar{v})}$ are holomorphic as well. This *weird* choice is made to let the final result compatible (and essentially equal), with the one in [14]. Coming back to our computations we get,

$$ujf(\alpha - i\beta) = u(\overline{G(\bar{v})}j - \overline{H(\bar{v})}),$$

and since $v \in \mathbb{C}^+$, then $\overline{G(\bar{v})} = \overline{\hat{g}(\bar{v})} = \hat{g}(v)$ (and analogously for H), hence,

$$(7) \quad \begin{aligned} \circledast &= [1 + uj, g(v) + h(v)j - u\hat{h}(v) + u\hat{g}(v)j] \\ &= \pi[1, u, g(v) - u\hat{h}(v), h(v) + u\hat{g}(v)], \end{aligned}$$

and so the lift coincide with the one computed in [14].

Finally, observe that if a slice regular function is defined over a domain which intersects the real line, then (as implicitly stated in [18]), $\hat{g} = \overline{g(\bar{v})}$ and analogously for h .

Remark 10. It will be useful to notice that the twistor lift of a slice regular function is always a rational map over its image.

Given the Representation formula for slice functions, exhibiting a slice function is equivalent to exhibit its defining stem function or its splitting over a complex plane \mathbb{C}_I for some $I \in \mathbb{S}$. In fact, in the next proofs and constructions we will define G and H starting from equation 7. In particular, given a slice regular function $f : \Omega_D \rightarrow \mathbb{H}$ which splits over D_i as $f(v) = G(v) + H(v)j$, it holds:

$$\begin{aligned} f(\alpha + I\beta) &= \frac{1}{2}[f(v) + f(\bar{v}) - Ii(f(v) - f(\bar{v}))] \\ &= \frac{1}{2}[G(v) + H(v)j + G(\bar{v}) + H(\bar{v})j - Ii(G(v) + H(v)j - G(\bar{v}) - H(\bar{v})j)] \\ &= \frac{1}{2}[(1 - Ii)(g(v) + h(v)j) + (1 + Ii)(\overline{\hat{g}(v)} + \overline{\hat{h}(v)}j)], \end{aligned}$$

where $\alpha + I\beta \in \Omega_D \setminus \mathbb{R}$ and $v = \alpha + i\beta$, with $\beta > 0$.

Given a slice regular function f we will say that its twistor lift \tilde{f} lies on a certain variety \mathcal{S} if the image of \tilde{f} is contained in \mathcal{S} .

4.1. Planes. Here we will show that, given a hyperplane in \mathbb{CP}^3 , then, the only non-constant slice regular functions that arise in our constructions are functions that do not extend to the real line. Take in fact a generic hyperplane given by the equation:

$$c_0X_0 + c_1X_1 + c_2X_2 + c_3X_3 = 0.$$

Substituting the coordinates in equation 7 in the previous equation we get

$$c_0 + c_1u + c_2(g(v) - u\hat{h}(v)) + c_3(h(v) + u\hat{g}(v)) = 0.$$

The left hand side of the last equation is, of course, a linear polynomial in u , so, the equality holds if and only if the next system is satisfied,

$$(8) \quad \begin{cases} c_0 + c_2g(v) + c_3h(v) = 0 \\ c_1 - c_2\hat{h}(v) + c_3\hat{g}(v) = 0. \end{cases}$$

Due to the nature of the lift we have to suppose that at least one between c_2 and c_3 is different from zero. Say, then $c_3 \neq 0$ (the case $c_2 \neq 0$ is obviously symmetric). Then, the last system become,

$$\begin{cases} h(v) = -c_3^{-1}(c_0 + c_2g(v)) \\ \hat{g}(v) = -c_3^{-1}(c_1 - c_2\hat{h}(v)). \end{cases}$$

Using now the representation formula 4, we can define the following slice regular function $f : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$, as

$$f(\alpha + I\beta) = \frac{(1 - Ii)}{2}(g(v) - c_3^{-1}(c_0 + c_2g(v))j) + \frac{(1 + Ii)}{2}(\overline{-c_3^{-1}(c_1 - c_2\hat{h}(v))} + \overline{\hat{h}(v)}j).$$

If this function extends to \mathbb{R} , then it must hold that $\hat{g}(v) = \overline{\hat{g}(\bar{v})}$ and $\hat{h}(v) = \overline{\hat{h}(\bar{v})}$. But if this is true, then from the system in equation 8 we get,

$$\hat{h}(v) = \overline{-c_3^{-1}c_0} + \overline{-c_3^{-1}c_2\hat{g}(v)},$$

and substituting this in the second equation we obtain that \hat{g} is equal to some constant complex number. Hence, the only way to obtain a slice regular function that extends to the real line is to take a constant function.

4.2. Quadrics. Thanks only to the general shape of the lift given in equation 6, we are able to prove the following result.

Theorem 26. *Let $f : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$ be a slice regular function. Then its twistor lift lies over the quadric in equation 5 if and only if f is a real slice function.*

Proof. Since the parametrization of the lift \tilde{f} is given by equation 6, then the condition of lying on the quadric 5 is encoded by the following system of equations

$$\begin{aligned} (9) \quad & p_i \circ F^{\perp c} = 0 = p_i \circ F^{\perp} \\ (10) \quad & p_i \circ F^{\top c} = p_i \circ F^{\top}, \end{aligned}$$

and so the slice regular function f with lifting equal to \tilde{f} can be constructed, thanks to equation 9, to be equal to,

$$f(\alpha + I\beta) = f^{\top}(\alpha + I\beta) = F_1^{\top}(\alpha + i\beta) + IF_2^{\top}(\alpha + i\beta).$$

But, thanks to equation 10 we have that

$$F_1^{\top}(\alpha + i\beta) + IF_2^{\top}(\alpha + i\beta) = F_1^{\top c}(\alpha + i\beta) + IF_2^{\top c}(\alpha + i\beta),$$

which implies that both $F_{1,2}^{\top}$ are real functions and so f is real.

The converse is trivial. \square

Now, the next result states that every non-singular quadric in the previous classification (see Theorem 4), can be reached by the lift of a slice regular function.

Theorem 27. *For any non-singular quadric in the classification of Theorem 4 there is an equivalent one \mathcal{Q} such that there exists a slice regular function f defined on a dense subset of $\mathbb{H} \setminus \mathbb{R}$, such that its twistor lift lies in \mathcal{Q} .*

Proof. For all the cases we will show the thesis exhibiting the splitting of f .

(1) If \mathcal{Q} is given as in equation 1, then it translates in set of solutions of

$$e^{\lambda+i\nu} + e^{-\lambda+i\nu}u^2 + e^{\mu-i\nu}(g(v) - u\hat{h}(v))^2 + e^{-\mu-i\nu}(h(v) + u\hat{g}(v))^2 = 0.$$

Writing the previous equation as a polynomial in u and imposing the vanishing of the coefficients we obtain the following system

$$\begin{cases} e^{\lambda+i\nu} + e^{\mu-i\nu}g^2 + e^{-\mu-i\nu}h^2 = 0 \\ -e^{\mu}g\hat{h} + e^{-\mu}h\hat{g} = 0 \\ e^{-\lambda+i\nu} + e^{\mu-i\nu}\hat{h}^2 + e^{-\mu-i\nu}\hat{g}^2 = 0 \end{cases}$$

From the first and the last equations we obtain

$$h^2 = -e^{\mu+i\nu}(e^{\lambda+i\nu} + e^{\mu-i\nu}g^2), \quad \hat{h}^2 = -e^{-\mu+i\nu}(e^{-\lambda+i\nu} + e^{-\mu-i\nu}\hat{g}^2).$$

Take now the square of second equation e substitute the values of h^2 and \hat{h}^2 :

$$e^{\mu}g^2(e^{-\lambda+i\nu} + e^{\mu-i\nu}\hat{g}^2) = e^{-\mu}\hat{g}^2(e^{\lambda+i\nu} + e^{\mu-i\nu}g^2),$$

that is

$$\hat{g} = \pm e^{\mu-\nu}g.$$

Taking now, for instance, $g(v) = v$, $\hat{g}(v) = e^{\mu-\nu}v$, $h = i(e^{\mu+i\nu}(e^{\lambda+i\nu} + e^{\mu-i\nu}g^2))^{1/2}$ and $\hat{h} = i(e^{-\mu+i\nu}(e^{-\lambda+i\nu} + e^{-\mu-i\nu}\hat{g}^2))^{1/2}$, we get the thesis in the first case.

(2) The last case is when \mathcal{Q} is the zero locus of the polynomial in 2 with $k \in [0, 1)$. Imposing then the usual equations we obtain that $g, h : \mathbb{C}_i \setminus \mathbb{R} \rightarrow \mathbb{C}$ and $\hat{g}, \hat{h} : \mathbb{C}_i^- \setminus \mathbb{R} \rightarrow \mathbb{C}$ can be chosen as

$$g(v) = -\hat{g}(v) = v, \quad h(v) = 2i + v/2, \quad \hat{h}(v) = 2i - v/2.$$

It is now a matter of computation, using the Representation formula, to write the slice regular functions defined by the previous three cases. \square

Remark 11. In the next section we will compute the points in \mathbb{S}^4 where the eventual twistor lines lie.

In the next theorem we will show that the result in Theorem 26 exhausts the set of non-singular algebraic surfaces (up to projective transformations) of degree 2, that can be reached by the twistor lift of a slice regular function. Some suspects that a result of this kind must hold came from the fact that there aren't *dominant rational maps*³ from \mathcal{Q} to any smooth varieties of degree $d \geq 4$. In fact, any smooth quadric in \mathbb{CP}^3 is projectively isomorphic to \mathcal{Q} (see, for instance, section 4 of [24]). Now, if $X \rightarrow Y$ is a dominant rational map between non-singular varieties in \mathbb{CP}^3 , then $\dim H^0(Y, K_Y) \leq \dim H^0(X, K_X)$, where K_X and K_Y stands for the canonical bundle of the subscript variety (see chapter 2, Section 8 of [25]). But $\dim H^0(\mathcal{S}, K_{\mathcal{S}})$ is greater or equal to 1 when the degree of \mathcal{S} is greater or equal to 4 and it is 0 when $d = 2, 3$.

Anyway the specific statement and proof follow.

Theorem 28. *Let \mathcal{S} be a non-singular algebraic surface of degree $d \geq 2$ in \mathbb{CP}^3 and let $\tilde{f} : \mathcal{Q}^+ \rightarrow \mathcal{S}$ the twistor lift of a slice regular function and such that $\tilde{f}(\mathcal{Q}^+)$ is open in \mathcal{S} . Then \mathcal{S} is projectively equivalent to \mathcal{Q} .*

Proof. Observe that for each fixed v_0 in \mathbb{CP}^1 , the twistor lift \tilde{f} of a generic slice regular function f , contains the whole line $l_{v_0} : \mathbb{CP}^1 \rightarrow \mathbb{CP}^3$ parametrized by $u \in \mathbb{CP}^1$. In formula

$$l_{v_0}[1, u] = [1, u, f^{\top}(v_0) - u f^{\perp c}(v_0), f^{\perp}(v_0) + u f^{\top c}(v_0)].$$

This is enough to prove the theorem since, from general facts about projective surfaces, we know that the number of lines over a non-singular surface of degree greater or equal to 3 in \mathbb{CP}^3 is always finite⁴. \square

Remark 12. The theory of lines or, in general, of rational curves over a surface is a very interesting and studied field. In particular we point out that several further information are stated about the nature of rational curves that can lie over a surface. Among the others we found Theorem 1.1 in [6] and Theorem 1 in [32], in which the authors state general formulas that implies that surfaces of degree greater or equal to 5 contain no lines. For the case in which the degree is equal to 3 we refer to [12] in which there is a summary of the whole story concerning the 27 lines over a cubic surface, while for degree equal to four we cite the classical paper [27] by Segre in which it is stated a upper bound on the number of lines over a quartic surface.

Remark 13. The case studied in [14] gave rise to a quartic ruled surface and so it is coherent with our last result.

After the last result one can search for classes of singular varieties that can be reached by the twistor lift of a slice regular function. Of course, since the argument of the proof is general, one can exclude from this classification all the surfaces which are not ruled by lines. And so, we obtain the following theorems.

Theorem 29. *Up to projective transformation any quadric surface $\mathcal{Q} \subset \mathbb{CP}^3$ is such that there exists a slice regular function f such that its twistor lift \tilde{f} lies on \mathcal{Q} .*

In the proof of this theorem, we will choose a particular union of two planes and a particular cone. Since the classification is projective this is enough to complete all the possible cases. If one is interested in singular quadric surfaces defined by different equations it may be possible to find no slice regular function whose lift realizes the chosen equation.

Proof. The smooth case is solved thanks to Theorem 26 and by the fact that all non-singular quadric are projectively equivalent. Up to projective transformations there are only two classes of singular quadric surfaces: the union of two planes and cones. We will show that there is a cone and a union of two planes that can be described with coordinates in accordance with equation 7.

³Meaning a rational map with dense image.

⁴ We would like to thanks Prof. Edoardo Ballico for the useful discussion about classical algebraic geometry.

- (1) Let \mathcal{P} be the union of two planes defined by the following equation

$$X_0^2 - X_2^2 = 0.$$

The slice regular function $f : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$ defined by $f(\alpha + I\beta) = (\alpha + I\beta)(1 - Ii)\frac{j}{2}$ lifts as $\tilde{f}[1, u, v, uv] = [1, u, 1, v]$ and so lies in \mathcal{P} .

- (2) Let \mathcal{K} be the quadratic cone defined by the following equation

$$X_1^2 = X_2X_3$$

Imposing then the usual equations we obtain that $G, H : \mathbb{C}_i \setminus \mathbb{R} \rightarrow \mathbb{C}_i$ can be chosen as

$$G(v) = \begin{cases} 0 & \text{if } v \in \mathbb{C}_i^+ \\ v & \text{if } v \in \mathbb{C}_i^- \end{cases}, \quad H(v) = \begin{cases} 0 & \text{if } v \in \mathbb{C}_i^+ \\ -\frac{1}{v} & \text{if } v \in \mathbb{C}_i^- \end{cases}.$$

As before, it is now a matter of computation, using the Representation formula, to write the slice regular functions defined by the previous equation. \square

4.3. Cubics. We will treat now the case of cubics surfaces. Firstly we will consider *non-normal* cubics and then cones. An algebraic variety X is said to be *normal* if it is normal at every point, meaning that the local ring at any point is an integrally closed domain. If X is a non-normal cubic surface, then its singular locus contains a 1 dimensional part (see [11], chapter 9.2).

Theorem 30. *Let \mathcal{C} be a non-normal cubic surface in \mathbb{CP}^3 that is not a cone. Then, up to projective isomorphisms, there exists a slice regular function f such that its twistor lift \tilde{f} lies on \mathcal{C} .*

Proof. In Theorem 9.2.1 of [11], the author says that, up to projective isomorphisms, the only non-normal cubic surfaces in \mathbb{CP}^3 that are not cones are the following two:

- (1) $X_0X_3^2 + X_1^2X_2 = 0$,
- (2) $X_0X_1X_3 + X_2X_3^2 + X_1^3 = 0$.

Putting the coordinates of the lift in Remark 9 in the previous equations we obtain, respectively,

- (1) $g(v) = -v^2$, $\hat{g}(v) = v$ and $h \equiv 0 \equiv \hat{h}$
- (2) $g(v) = -1/v$, $\hat{g}(v) = v$, $\hat{h}(v) = 1/v^2$ and $h \equiv 0$

and so, if we put $x = \alpha + I\beta$ and $v = \alpha + i\beta$, the two slice regular functions are, respectively,

- (1) $f_1 : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$ defined by

$$(11) \quad f_1(x) = -x^2 \frac{(1 - Ii)}{2} + x \frac{(1 + Ii)}{2},$$

- (2) $f_2 : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$ defined by

$$f_2(x) = -x^{-1} \frac{(1 - Ii)}{2} + x \frac{(1 + Ii)}{2} + x^{-2} \frac{(1 + Ii)}{2} j$$

\square

The last case that we will treat is the case of cubic cones. The set of cubic cones can be defined by the equation

$$(12) \quad X_3^3 - (c + 1)X_3^2X_1 + cX_3X_1^2 - X_2^2X_1 = 0,$$

where, if $c \in \mathbb{C} \setminus \{0, 1\}$, the surface is a cone over a non-singular plane cubic curve, while, in the case in which $c = 0, 1$ the surface is a cone over a nodal or cuspidal plane cubic curve respectively.

Theorem 31. *Let \mathcal{C} be cubic cone. Then there exist a slice regular function f defined on a dense subset of $\mathbb{H} \setminus \mathbb{R}$, such that, up to projective transformations, its twistor lift \tilde{f} lies on \mathcal{C} .*

Proof. As in the previous theorems we will prove this result by exhibiting the splitting of the function f . If we impose equation 12 in the coordinates 7 we obtain that g and h must be identically zero while \hat{g} and \hat{h} must satisfy the following equation

$$\hat{g}^3 - (c + 1)\hat{g}^2 + c\hat{g} = \hat{h}^2.$$

Solving then in \hat{h} or in \hat{g} , one finds the desired splitting of the slice regular function that give the thesis. \square

Since, up to projective transformations, the only cubic surfaces that contain infinite lines are cones and the non-normal ones, then, the projective classification is complete.

Of course, the functions seen in the previous proofs are not the only slice regular functions that solve the problem and give the thesis. One could ask for the “best” slice regular function such that its lift satisfies a certain algebraic equation, but this issue will not be treated in this paper and we propose it for some future work.

5. RATIONAL CURVES ON THE GRASSMANNIAN

The aim of this section is to reconstruct the *twistor transform* defined in [14] for slice regular functions that are not defined on the real line. Moreover at the end we will characterize certain rational curves over the Grassmannian $\mathbb{G}r_2(\mathbb{C}^4)$.

The non-singular quadric in equation 5 is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the rulings are parametrized by u and v . A sphere $\alpha + \mathbb{S}\beta$ can be identified with the line,

$$l_{v_0} := \{[1, u, \alpha + i\beta, (\alpha + i\beta)u] \mid u \in \mathbb{C} \cup \{\infty\}\} \subset \mathbb{CP}^3,$$

defined by fixing $v_0 = \alpha + i\beta$. The line l_{v_0} can also be seen as a point in the Grassmannian $\mathbb{G}r_2(\mathbb{C}^4)$ or, equivalently, as a point in the Klein quadric in $\mathbb{P}(\wedge^2 \mathbb{C}^4) \simeq \mathbb{CP}^5$ via Plücker embedding.

As we seen in section 2, left multiplication by j on \mathbb{H}^2 lifts in \mathbb{C}^4 as

$$[X_0, X_1, X_2, X_3] \xrightarrow{j} [-\overline{X_1}, \overline{X_0}, -\overline{X_3}, \overline{X_2}],$$

and the last induces a real structure σ over \mathbb{CP}^5 as follows,

$$\sigma : [\xi_1, \dots, \xi_6] \mapsto [\bar{\xi}_1, \bar{\xi}_5, -\bar{\xi}_4, -\bar{\xi}_3, \bar{\xi}_2, \bar{\xi}_6],$$

where $\{\xi_1, \dots, \xi_6\}$ represent the basis $\{e^{01}, e^{02}, e^{03}, e^{12}, e^{13}, e^{23}\}$ of $\wedge^2 \mathbb{C}^4$ and, of course, $e^{ij} := e^i \wedge e^j$. In the above coordinates we can explicit the equation of the Klein quadric as follows,

$$(13) \quad \xi_1 \xi_6 - \xi_2 \xi_5 + \xi_3 \xi_4 = 0.$$

As explained previously in section 3 (and in section 2 of [28]), a *fixed point of σ corresponds to a j -invariant line in \mathbb{CP}^3 , i.e. a (twistor) fibre of π .*

Example 3. Consider the coordinates founded in Theorem 27 as functions defined on $\mathbb{CP}^1 \times \mathbb{CP}^1$. We want to find the twistor fibre mentioned in the previous result imposing equation $\sigma(\mathcal{F}(v)) = \mathcal{F}(v)$.

- (1) If $\lambda = \mu \neq 0$ and $\nu = \pi/2$ we get, $\mathcal{F} : v \mapsto [1, c(1 - v^2)^{1/2}, -v, v, \frac{1}{c}(1 - v^2)^{1/2}, 1]$. Imposing $\sigma(\mathcal{F}(v)) = \mathcal{F}(v)$, we obtain $v = \pm 1$ (i.e. *two twistor lines* in correspondence of $x = \pm 1 \in \mathbb{R}$).
- (2) If $\lambda = \mu = 0$ and $\nu \in (0, \pi/2)/2$ we get,

$$\mathcal{F} : v \mapsto [v^2 - \frac{e^{2i\nu} + v^2}{\|e^{i\nu}\|^2}, \frac{i}{\|e^{i\nu}\|}(e^{2i\nu} + v^2)^{1/2}, -v, v, \frac{i}{\|e^{i\nu}\|}(e^{2i\nu} + v^2)^{1/2}, 1].$$

Imposing $\sigma(\mathcal{F}(v)) = \mathcal{F}(v)$, we obtain no solution or *no twistor lines* (this because ω is a fixed non-real complex number).

- (3) If \mathcal{Q} is the zero set of the polynomial in equation 2, we get, $\mathcal{F} : v \mapsto [-(\frac{5}{4}v^2 + 4), 2i + \frac{v}{2}, -v, -v, 2i - \frac{v}{2}, 1]$. Imposing $\sigma(\mathcal{F}(v)) = \mathcal{F}(v)$, we obtain $v = -4i$ (i.e. *one twistor line* in correspondence of $x = -4i \in \mathbb{H}$).

At this point we can extend the definition given in [14] of twistor transform.

Definition 17 (Twistor transform). Let $D \subset \mathbb{C}^+$ be a domain and $f : \Omega_D \rightarrow \mathbb{H}$ be a slice function. We define its *twistor transform* of f as the following map:

$$\begin{aligned} \mathcal{F} : D &\rightarrow \mathbb{G}r_2(\mathbb{C}^4) \\ v &\mapsto \tilde{f}(l_v). \end{aligned}$$

The following result extends Theorem 5.7 of [14].

Theorem 32. *Let D be a domain in \mathbb{C}^+ . If $f : \Omega_D \rightarrow \mathbb{H}$ is a slice function, then its twistor transform \mathcal{F} defines a curve over D . Moreover, every curve $\gamma : D \rightarrow \mathbb{G}r(\mathbb{C}^4)$, such that $\xi_6 \circ \gamma$ is never zero, is the twistor transform of a slice function $f : \Omega_D \rightarrow \mathbb{H}$. The function f is regular if and only if its twistor transform is a holomorphic curve.*

Proof. Given a slice function $f : \Omega_D \rightarrow \mathbb{H}$, its twistor lift is given, as in 6, by, $\tilde{f}[1, u, v, uv] = [1, u, p_i \circ F^\top(v) - u(p_i \circ F^{\perp c}(v)), p_i \circ F^\perp(v) + u(p_i \circ F^{\top c}(v))]$, where f^\top and f^\perp are the same as in formula 6. Fixing v , $\tilde{f}(l_v)$ is defined by the following linear equations:

$$\begin{cases} X_0(p_i \circ F^\top) - X_1(p_i \circ F^{\perp c}) - X_2 = 0 \\ X_0(p_i \circ F^\perp) + X_1(p_i \circ F^{\top c}) - X_3 = 0. \end{cases}$$

The coefficients of the last two equations determines the following generating vectors

$$e_1 = [p_i \circ F^\top, -p_i \circ F^{\perp c}, -1, 0], \quad e_2 = [p_i \circ F^\perp, p_i \circ F^{\top c}, 0, -1].$$

Using equation 13, then, the twistor transform can be made explicit as follows

$$\begin{aligned} \mathcal{F}(v) = [\xi_1, \dots, \xi_6] = & [(p_i \circ F^\top)(v)(p_i \circ F^{\top c})(v) + (p_i \circ F^\perp)(v)(p_i \circ F^{\perp c})(v), \\ & (p_i \circ F^\perp)(v), -(p_i \circ F^\top)(v), (p_i \circ F^{\top c})(v), (p_i \circ F^{\perp c})(v), 1], \end{aligned}$$

where $\{\xi_i\} = \{e_1^h \wedge e_2^k\}_{0 \leq h < k \leq 3}$. But now that we have the explicit parametrization of $\mathcal{F}(v)$ it is clear that this is a holomorphic curve if and only if f is a slice regular function.

Vice versa, given a curve $\gamma : D \rightarrow \mathbb{G}r_2(\mathbb{C}^4)$ such that $\xi_6 \circ \gamma$ is never zero, we can assume $\xi_6 \circ \gamma = 1$ and recover the splittings of f as follows,

$$(p_i \circ F^\top) = -\xi_3 \circ \gamma, \quad (p_i \circ F^\perp) = \xi_2 \circ \gamma, \quad (p_i \circ F^{\top c}) = \xi_4 \circ \gamma, \quad (p_i \circ F^{\perp c}) = \xi_5 \circ \gamma.$$

Thanks to the Representation Theorem we can now recover f . Then, thanks to remark 3 we obtain regularity. \square

From the proof, then, came out that the twistor transform \mathcal{F} of a slice regular function f , can be represented in the following way,

$$\begin{aligned} \mathcal{F}(v) = & [(p_i \circ F^\top)(v)(p_i \circ F^{\top c})(v) + (p_i \circ F^\perp)(v)(p_i \circ F^{\perp c})(v), \\ & (p_i \circ F^\perp)(v), -(p_i \circ F^\top)(v), (p_i \circ F^{\top c})(v), (p_i \circ F^{\perp c})(v), 1]. \end{aligned}$$

Remark 14. As for Theorem 24, in the last theorem we could repeat the computations using the splitting lemma. The result would be the following,

$$\mathcal{F}(v) = [g(v)\hat{g}(v) + \hat{h}(v)h(v), h(v), -g(v), \hat{g}(v), \hat{h}(v), 1],$$

which coincide with the result in [14].

We will now present some examples.

- Example 4.** • Let $f_1 : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$ be the following slice regular function: $f(\alpha + I\beta) = 1 - Ii$. This function is equal to 2 over \mathbb{C}_i and to 0 over \mathbb{C}_{-i} . Its twistor transform $\mathcal{F}_1 : \mathbb{C}^+ \rightarrow \mathbb{G}r(\mathbb{C}^4)$ is the constant function $v \mapsto [0, 0, -2, 0, 0, 1]$.
- Let $f_2 : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$ be the following slice regular function: $f(\alpha + I\beta) = 1 + Ii$. This function is equal to 0 over \mathbb{C}_i and to 2 over \mathbb{C}_{-i} . Its twistor transform $\mathcal{F}_2 : \mathbb{C}^+ \rightarrow \mathbb{G}r(\mathbb{C}^4)$ is the constant function $v \mapsto [0, 0, 0, 2, 0, 1]$.
 - Let $f_3 : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$ be the following slice regular function: $f(\alpha + I\beta) = (\alpha + I\beta)(1 - Ii)/2$. This function is equal to $(\alpha + I\beta)$ over \mathbb{C}_i and to 0 over \mathbb{C}_{-i} . Its twistor transform $\mathcal{F}_3 : \mathbb{C}^+ \rightarrow \mathbb{G}r(\mathbb{C}^4)$ is the function $v \mapsto [0, 0, -v, 0, 0, 1]$.
 - Let $f_4 : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$ be the following slice regular function: $f(\alpha + I\beta) = (\alpha + I\beta)(1 + Ii)/2$. This function is equal to 0 over \mathbb{C}_i and to $(\alpha + I\beta)$ over \mathbb{C}_{-i} . Its twistor transform $\mathcal{F}_4 : \mathbb{C}^+ \rightarrow \mathbb{G}r(\mathbb{C}^4)$ is the function $v \mapsto [0, 0, 0, v, 0, 1]$.

As said at the beginning of this section we want to characterize a certain set of linear holomorphic functions $\gamma : D \rightarrow \mathbb{G}r(\mathbb{C}^4)$ in terms of slice regular functions. We will restrict to the case in which $\xi_6 \circ \gamma$ is never zero, The theorem we are going to prove is the following.

Theorem 33. *Let $\gamma : \mathbb{C}^+ \rightarrow \mathbb{G}r(\mathbb{C}^4)$ be a holomorphic curve such that $\xi_6 \circ \gamma$ is never zero. Then γ is affine if and only if there exist $A, B \in \mathbb{C}$, with $A/B \in \mathbb{C}^+ \cup \mathbb{R}$ such that γ is the twistor transform of a slice regular function f and $(A + xB) \cdot f$ is a slice affine function that satisfies*

$$(14) \quad h_i(Af_i - Bg_i, \bar{A}f_{-i} - \bar{B}g_{-i}) = 0,$$

where $f_{\pm i}$ are the values of the slice derivative of $(A + xB) \cdot f$ in $\mathbb{C}_{\pm i}$, $g_{\pm i}$ are the values of the slice constant function $(A + xB) \cdot f - x[(1 - Ii)f_i + (1 + Ii)f_{-i}]$ in $\mathbb{C}_{\pm i}$ and h_i denotes the hermitian product in $\mathbb{C}_i \oplus \mathbb{C}_i^\perp \simeq \mathbb{H}$.

Proof. A linear map $\gamma : \mathbb{C}^+ \rightarrow \mathbb{G}r(\mathbb{C}^4)$ is a map of the form,

$$\gamma(v) = [c_{11} + c_{12}v, c_{21} + c_{22}v, c_{31} + c_{32}v, c_{41} + c_{42}v, c_{51} + c_{52}v, c_{61} + c_{62}v],$$

intending the Grassmannian $\mathbb{G}r_2(\mathbb{C}^4)$ as the Klein quadric 13 in \mathbb{CP}^5 . The condition $\xi_6 \circ \gamma \neq 0$ for all $v \in \mathbb{C}^+$ can be interpreted, of course, as $c_{61}/c_{62} \in \mathbb{C}^+ \cup \mathbb{R}$. Dividing everything by $c_{61} + c_{62}v$, we obtain

$$\gamma(v) = \left[\frac{c_{11} + c_{12}v}{c_{61} + c_{62}v}, \frac{c_{21} + c_{22}v}{c_{61} + c_{62}v}, \frac{c_{31} + c_{32}v}{c_{61} + c_{62}v}, \frac{c_{41} + c_{42}v}{c_{61} + c_{62}v}, \frac{c_{51} + c_{52}v}{c_{61} + c_{62}v}, 1 \right],$$

and so, now $\xi_6 \circ \gamma = 1$. Substituting then the components of γ in equation 13, one obtain the following system of equations:

$$(15) \quad \begin{cases} c_{11}c_{61} - c_{21}c_{51} + c_{31}c_{41} = 0 \\ c_{11}c_{62} + c_{12}c_{61} - (c_{21}c_{52} + c_{22}c_{51}) + (c_{31}c_{42} + c_{32}c_{41}) = 0 \\ c_{12}c_{62} + c_{32}c_{42} + c_{22}c_{52} = 0 \end{cases}.$$

Moreover, since γ is a holomorphic function, then it will be the twistor transform of some slice regular function f such that

$$\begin{aligned} f_{\mathbb{C}_i^+}(\alpha + i\beta) &= -\frac{c_{31} + c_{32}(\alpha + i\beta)}{c_{61} + c_{62}(\alpha + i\beta)} + \frac{c_{21} + c_{22}(\alpha + i\beta)}{c_{61} + c_{62}(\alpha + i\beta)}j \\ f_{\mathbb{C}_{-i}^+}(\alpha - i\beta) &= \frac{\overline{c_{41} + c_{42}(\alpha + i\beta)}}{c_{61} + c_{62}(\alpha + i\beta)} + \frac{\overline{c_{51} + c_{52}(\alpha + i\beta)}}{c_{61} + c_{62}(\alpha + i\beta)}j. \end{aligned}$$

With the Representation Formula one obtain that, for each $\alpha + I\beta \in \mathbb{H} \setminus \mathbb{R}$,

$$\begin{aligned} 2f(\alpha + I\beta) &= [(1 - Ii)f(\alpha + i\beta) + (1 + Ii)f(\alpha - Ii)] \\ &= (c_{61} + (\alpha + I\beta)c_{62})^- \cdot [(\alpha + I\beta)(1 - Ii)(-c_{32} + c_{22}j) + (1 - Ii)(-c_{31} + c_{21}j)] \\ &\quad + (c_{61} + (\alpha + I\beta)c_{62})^- \cdot [(\alpha + I\beta)(1 + Ii)(\bar{c}_{42} + \bar{c}_{52}j) + (1 + Ii)(\bar{c}_{41} + \bar{c}_{51}j)], \end{aligned}$$

but then, $(c_{61} + (\alpha + I\beta)c_{62}) \cdot f$ is a slice affine function. If now, one between c_{61} or c_{62} is equal to zero this correspond, respectively, to A or B equal to zero and so equation 14 holds true. If both c_{61} and c_{62} are non-zero, observe that, the first and the third equations in 15 can be written, respectively, as $h_i(g_i, g_{-i}) = c_{11}A$ and $h_i(f_i, f_{-i}) = c_{12}B$. Substituting these in the second equation of the system and since $(c_{21}c_{52} + c_{22}c_{51}) - (c_{31}c_{42} + c_{32}c_{41}) = h_i(g_i, f_{-i}) + h_i(f_i, g_{-i})$, we get

$$h_i(g_i, g_{-i})\frac{B}{A} + h_i(f_i, f_{-i})\frac{A}{B} = h_i(g_i, f_{-i}) + h_i(f_i, g_{-i}),$$

and so equation 14 holds true. The vice versa is trivial, following the proof in the opposite verse. \square

Example 5. Simple examples of slice regular functions that satisfies the condition in equation 14, are all the functions of the following type:

$$\begin{aligned} f : \quad \mathbb{H} \setminus \mathbb{R} &\rightarrow \mathbb{H} \\ \alpha + I\beta &\mapsto (Cx + D)^- \cdot (Ax + B)(1 - Ii)/2, \end{aligned}$$

with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2, \mathbb{R})$. In the next section we will explore one particular function in this set and then we will add some remarks to the whole family.

Remark 15. The set of slice affine functions that satisfy 14 does not contain non constant slice functions that extend to the real line. In fact, as shown in remark 7, a slice affine function extends to \mathbb{R} if the coefficients of first order are equal, i.e.: $f_+ = f_-$, meaning that $h_i(f_i, f_{-i}) \neq 0$.

6. A FIRST NON TRIVIAL EXAMPLE

In this section we will study the following slice regular function

$$(16) \quad \begin{aligned} f : \mathbb{H} \setminus \mathbb{R} &\rightarrow \mathbb{H} \\ \alpha + I\beta &\mapsto (\alpha + I\beta)(1 - Ii)/2 \end{aligned}$$

as tool to generate OCSes over its image. We will write also, for brevity, $f(x) = x(1 - Ii)/2$, where $x = \alpha + I\beta \in \mathbb{H} \setminus \mathbb{R}$. As was shown in [2], this function is constant and equal to 0 if restricted to \mathbb{C}_{-i}^+ and equal to x if restricted to \mathbb{C}_i^+ . In the same section was shown either theoretically and by explicit computations that the restriction to $\mathbb{H} \setminus \mathbb{C}_{-i}^+$ is an open function. In [3] was proved that, if restricted to $\mathbb{H} \setminus \mathbb{C}_{-i}^+$, the function f is injective. For these reasons this function fit very well in the twistorial construction studied here. Moreover, this construction has a symbiotic aspect with respect to the function f . In fact, with the help of the twistor lift stated in Theorem 24 it is possible to understand constructively the image of f . The next theorem precise this fact.

Theorem 34. *If $q = q_0 + q_1i + q_2j + q_3k$, then the function defined in equation 16 is such that $f(\mathbb{H} \setminus \mathbb{C}_{-i}^+) = \{q \in \mathbb{H} \mid q_1 > 0\}$. Moreover*

$$\bigcup_{I \in \mathbb{S}} f|_{\mathbb{C}_I^+}(\mathbb{R}) = \{q \in \mathbb{H} \mid q_1 = 0\},$$

where $f|_{\mathbb{C}_I^+}(\mathbb{R})$ means the unique extension to \mathbb{R} of the function restricted to \mathbb{C}_I^+ .

Proof. To prove the theorem we will use the twistor lift 6. In fact, thanks to Theorem 24, it is possible to compute the image of a slice regular function by looking at the image of the projection to \mathbb{H} of its twistor lift. Since, as already said, the function f is equal to the identity if restricted to \mathbb{C}_i^+ and to zero over the opposite semislice \mathbb{C}_{-i}^+ , then its twistor lift is defined as follows:

$$(17) \quad \begin{aligned} F : \mathcal{Q}^+ \cap \pi^{-1}(\mathbb{H} \setminus \mathbb{C}_{-i}^+) &\rightarrow \mathbb{CP}^3 \\ [1, u, v, uv] &\mapsto [1, u, v, 0], \end{aligned}$$

where, if $\alpha + I\beta \in \mathbb{H} \setminus \mathbb{C}_i^+$ and $I = ai + bj + ck$, then $u = -i\frac{b+ic}{a+1}$ and $v = \alpha + i\beta$, with $(a, b, c) \neq (-1, 0, 0)$ and $\beta > 0$. At the end what we want to compute is the image of the function $(1 + uj)^{-1}v$ and so these are the computations:

$$\begin{aligned} (1 + uj)^{-1}v &= \left(1 - \frac{b+ic}{a+1}k\right)(\alpha + i\beta) \\ &= \frac{(a+1)^2}{(a+1)^2 + (b^2 + c^2)} \left(1 + \frac{bk - cj}{a+1}\right)(\alpha + i\beta) \\ &= \frac{1}{2}[(a+1)(\alpha + i\beta) + (\beta b - \alpha c)j + (\alpha b + \beta c)k]. \end{aligned}$$

So, the image of a quaternion $x = \alpha + (ai + bj + ck)\beta$ via f , with $ai + bc + ck \in \mathbb{S} \setminus \{-i\}$ and $\beta > 0$ is the quaternion

$$2f(x) = \alpha(a+1) + \beta(a+1)i + (\beta b - \alpha c)j + (\alpha b + \beta c)k.$$

Take now a generic quaternion $q = q_0 + q_1i + q_2j + q_3k$, this will be reached by f if and only if $q_1 > 0$. In fact the system

$$\begin{cases} \alpha(a+1) = q_0 \\ \beta(a+1) = q_1 \\ \beta b - \alpha c = q_2 \\ \alpha b + \beta c = q_3, \end{cases}$$

can be solved in the following way: the first two equations give $\alpha = q_0/(a+1)$ and $\beta = q_1/(a+1)$ and since $(a+1) \in (0, 2]$, then $q_1 > 0$. If we put $B = b/(a+1)$ and $C = c/(a+1)$, the last two equations can be rewritten as

$$\begin{cases} q_1 B - q_0 C = q_2 \\ q_0 C + q_1 B = q_3, \end{cases}$$

which is a linear system such that the two equations are linearly independent, so the solutions is,

$$B = \frac{q_1 q_2 + q_0 q_3}{q_0^2 + q_1^2}, \quad C = \frac{q_1 q_3 - q_0 q_2}{q_0^2 + q_1^2}.$$

Now we remember that $a^2 + b^2 + c^2 = 1$ and so $B^2 + C^2 = \frac{1-a}{1+a}$ that entails $a = \frac{1-B^2-C^2}{1+B^2+C^2}$ which is always an admissible solution since it is always different from -1 .

For the second part of the theorem, fix $I = ai + bj + ck \in \mathbb{S} \setminus \{-i\}$ and look for the following limit,

$$\lim_{\substack{\beta \rightarrow 0 \\ \alpha + I\beta \in \mathbb{C}^+}} f(\alpha + I\beta).$$

After restricting the function to \mathbb{C}_I^+ it is possible to extend it to \mathbb{R} and also to look at the image via the twistor lift. Since f is continuous we obtain that, up to a factor 2, the previous limit is equal to

$$\alpha(a+1) - \alpha c j + \alpha b k = \alpha(a+1, 0, -c, b),$$

which is a straight line belonging to the set $\{q \in \mathbb{H} \mid q_1 = 0\}$ passing through the vector $(a+1, 0, -c, b)$. Taking the union, for (a, b, c) that runs over $\mathbb{S} \setminus \{-i\}$, it is clear that this will span the whole hyperplane $\{q_1 = 0\}$. \square

The twistor lift of f lies in the hypersurface $\mathcal{H} := \{X_3 = 0\} \subset \mathbb{CP}^3$. In this case the general theory (see Section 3 of the present paper and Section 3 of [26]) says that \mathcal{H} induces an OCS conformally equivalent to a constant one, defined over the image of f . This is actually true and we will show that there is a specific conformal function from $\{q_1 > 0\} \subset \mathbb{H}$ to $\{q_1 < 0\}$ that sends \mathbb{J}^f to i . The theorem is the following one.

Theorem 35. *The complex metric manifold $(\{q_1 > 0\}, g_{Eucl}, \mathbb{J}^f)$ is conformally equivalent to $(\{q_1 < 0\}, g_{Eucl}, \mathbb{J}_i)$, where, with \mathbb{J}_i we mean the left multiplication by i . The conformality is determined by the function $g : \{q_1 > 0\} \rightarrow \{q_1 < 0\}$ defined by $g(q) = q^{-1}$.*

Proof. The function g is of course a conformal map for the Euclidean metric. So, the only thing to prove is that the push-forward of \mathbb{J}^f via g is exactly \mathbb{J}_i , meaning that, the following equality holds true

$$dg \circ \mathbb{J}^f = \mathbb{J}_i \circ dg.$$

But what is the actual shape of the two complex structures \mathbb{J}^f and \mathbb{J}_i ? The answer is easy and can be found analyzing the action against a generic tangent vector on a point. So we have that, if $v = (v_0, v_1, v_2, v_3)$ is a tangent vector over $p = f(\alpha + I\beta)$, then, $\mathbb{J}_i(p)v = (-v_1, v_0, -v_3, v_2)$, while $\mathbb{J}^f(p)v = (-av_1 - bv_2 - cv_3, av_0 - cv_2 + bv_3, bv_0 + cv_1 - av_3, cv_0 - bv_1 + av_2)$, where $ai + bj + ck = I$.

And so

$$\mathbb{J}_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbb{J}^f(p) = \begin{pmatrix} 0 & -a & -b & -c \\ a & 0 & -c & b \\ b & c & 0 & -a \\ c & -b & a & 0 \end{pmatrix},$$

where $p = p_0 + p_1 i + p_2 j + p_3 k$ (see also remark 1) and, working on the computations in the proof of Theorem 34,

$$a = \frac{p_0^2 + p_1^2 - p_2^2 - p_3^2}{|p|^2}, \quad b = 2 \frac{p_0 p_3 + p_1 p_2}{|p|^2}, \quad c = 2 \frac{p_1 p_3 - p_0 p_2}{|p|^2}.$$

Now, writing g as $g(q_0 + q_1i + q_2j + q_3k) = (q_0, -q_1, -q_2, -q_3)/|q|^2$, one have that

$$dg(q) = \begin{pmatrix} |q|^2 - 2q_0^2 & -2q_0q_1 & -2q_0q_2 & -2q_0q_3 \\ 2q_1q_0 & -|q|^2 + 2q_1^2 & 2q_1q_2 & 2q_1q_3 \\ 2q_2q_0 & 2q_2q_1 & -|q|^2 + 2q_2^2 & 2q_2q_3 \\ 2q_3q_0 & 2q_3q_1 & 2q_3q_2 & -|q|^2 + q_3^2 \end{pmatrix} / |q|^4$$

and that,

$$\mathbb{J}_i \circ dg = \begin{pmatrix} -2q_1q_0 & |q|^2 - 2q_1^2 & -2q_1q_2 & -2q_1q_3 \\ |q|^2 - 2q_0^2 & -2q_0q_1 & -2q_0q_2 & -2q_0q_3 \\ -2q_3q_0 & -2q_3q_1 & -2q_3q_2 & |q|^2 - 2q_3^2 \\ 2q_2q_0 & 2q_2q_1 & -|q|^2 + 2q_2^2 & 2q_2q_3 \end{pmatrix} / |q|^4.$$

It is now a matter of computation to show that $dg \circ \mathbb{J}^f = \mathbb{J}_i \circ dg$, but we will skip it. \square

The previous Theorem implies, in particular, the existence of a biholomorphism between the two complex manifolds $(\mathbb{H} \setminus \mathbb{C}_{-i}^+, \mathbb{J})$ and (\mathbb{C}^2, i) .

Remark 16. The function $g(q) = q^{-1}$ in the previous theorem, was found using the following idea. The constant OCS \mathbb{J}_i is described by the hyperplane $\{X_1 = 0\} \subset \mathbb{CP}^3$ (see Remark 2.3 of [26]) and so, starting from our lift $[1, u, v, 0]$ after changing the first two coordinates with the second two and dividing everything by $v(\neq 0)$, we obtain $[1, 0, v^{-1}, v^{-1}u]$ that projects to $[1, v^{-1}(1+uj)]$, but now $v^{-1}(1+uj) = ((1+uj)^{-1}v)^{-1} = (f(q))^{-1}$.

Remark 17. The last theorem and construction can be obtained using the following function as well: $f : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$, defined as

$$f(\alpha + I\beta) = (Cx + D)^{-1} \cdot (Ax + B) \frac{(1 - Ii)}{2},$$

with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2, \mathbb{R})$, $x = \alpha + I\beta$ and $z = \alpha + i\beta$. In fact, if we remove from the domain of this function the semislice \mathbb{C}_{-i}^+ over which is equal to zero, f is open and injective and its image is equal again to $\{q \in \mathbb{H} \mid q_1 > 0\}$. With easy computations one obtains that

$$f(\alpha + I\beta) = \begin{cases} \frac{(a+1)}{2\|Cz + D\|^2} [CA\|z\|^2 + DB + (BC + AD)\alpha] = q_0 \\ \frac{(a+1)}{2\|Cz + D\|^2} \beta = q_1 \\ \frac{(b\beta - c[CA\|z\|^2 + DB + (BC + AD)\alpha])}{2\|Cz + D\|^2} = q_2 \\ \frac{c\beta + b[CA\|z\|^2 + DB + (BC + AD)\alpha]}{2\|Cz + D\|^2} = q_3, \end{cases} \quad , \quad z = \alpha + i\beta,$$

and, with the same argument in the proof of Theorem 34, we obtain that $q_1 > 0$ and, for any values of q_0, q_1 , each q_2 and q_3 can be reached. Now, on the remaining first two components the function is exactly equal to

$$\frac{A(\alpha + i\beta) + B}{C(\alpha + i\beta) + D} = \frac{q_0 + iq_1}{(a+1)},$$

but, since A, B, C, D are taken such that the matrix they describe is in $SL(2, \mathbb{R})$, then since the function on the left describes an automorphism of the upper half complex space, it turns out that each q_0 and $q_1 > 0$ can be reached. The twistor lift of this function is

$$\begin{aligned} \tilde{f} : \mathcal{Q}^+ \cap \pi^{-1}(\mathbb{H} \setminus \mathbb{C}_{-i}^+) &\rightarrow \mathbb{CP}^3 \\ [1, u, v, uv] &\mapsto [1, u, \frac{Av+B}{Cv+D}, 0] \end{aligned}$$

In the next remark we will show an idea that we haven't explored completely but might be a starting point for some future considerations.

Remark 18. The twistor lift in equation 17, extends to a holomorphic mapping $\tilde{f} : \mathcal{Q} \rightarrow \mathbb{CP}^3$ by allowing v to take values in \mathbb{C} rather than just in \mathbb{C}^+ . However, even if $\pi \circ \tilde{f} = f \circ \pi$ on $\mathcal{Q}^+ \cap \pi^{-1}(\mathbb{H} \setminus \mathbb{C}_{-i}^+)$,

$$\begin{array}{ccc} \mathcal{Q}^+ & \xrightarrow{\tilde{f}} & \{X_3 = 0\} \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{H} \setminus \mathbb{R} & \xrightarrow{f} & \{q_1 > 0\} \end{array}$$

this does not imply that the graph will commute once \tilde{f} is extended. In fact we will have the following diagram,

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{1:1} & \{X_3 = 0\} \\ 2:1 \downarrow & & \downarrow 1:1 \\ ? & \xrightarrow{*1:2*} & ? \end{array}$$

where, the numbers upon the arrows are intended as *generically* and we don't know *a priori* what is in the two corners below and what is the meaning of the arrow that connects them. Also this arrow must represent something which behaves like $1:2$. This of course cannot be possible and suggest the possibility of approaching the issue using *multi-valued* functions. Anyway this example seems enough easy to be studied “by hands”. So, first of all, we need to construct the “ghost function” that realize the second part of that $1:2$ cited before. So, when we extend \tilde{f} to the whole \mathcal{Q} we need the function that realizes the lifting $\tilde{f}[1, u, v, uv] = [1, u, v, 0]$, for $v \in \mathbb{C}^-$.

In a certain sense, once you decompose the function in its four real components, the variable β and I are able to move completely freely. So, depending on the interpretation one give to the point $x(= \alpha + I\beta = \alpha + (-I)(-\beta))$, the representation of the function f in its vectorial form, returns two different values.

7. CONCLUSION AND FUTURE WORKS

In this paper, after a brief review of slice regular functions and twistor space of \mathbb{S}^4 , we have shown that the theory introduced in [14] linking these two fields, can be extended to all slice regular functions. Moreover, the techniques used to extend the theory of slice regular functions to domains with empty intersection with the real line were used to show a number of new results, such as the second part of Theorem 24. In this framework we have proved that this theory is effective in giving coordinates for the quadric surfaces in the conformal classification of non-singular quadrics in Theorem 4 and we gave a projective classification of the remaining quadrics and cubics that can be reach by the lift of a slice regular function. Finally we have used all this material to show the effectiveness of this instruments in the task of finding an explicit biholomorphism between two particular complex 4-manifolds.

With these results we hope to obtain further results in this direction. Some open problems that we would like to explore in the future (some of them are, actually, work in progress) regards the conformal classification of remaining (singular) quadrics and cubics. Strongly linked to this problem we plan to solve the ambiguous “multifunction” issue contained in the last remark. Furthermore we would like to study more the geometry of lines expressed by the twistor transform. In particular it would be interesting to classify other classes of rational curves over the Grassmannian.

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